Hybrid models with delay in the analysis of biophysical phenomena

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We are developing a computational modeling method for transient biophysical processes with crisis-like transformation effects under the influence of delay effects. During invasions of aggressive species in ecosystems and COVID-19 epidemics, alternating forms of oscillatory activity regimes emerge and decay. We simulate series of invasive waves using scenario-based equations with delays and redefinable oscillation damping functions. We construct simulation models of invasions from equations for stage transitions based on hybrid automata with events that determine the change in the evolutionary adaptation mode of stages of a decaying oscillatory biophysical process from increasing and decreasing waves of sharp population activity. The hybrid model allows us to describe the conditions for the transition of a crisis situation to a new stage in a series of waves of COVID-19 or aggressive invasion.

Keywords: hybrid automata structures, oscillating invasions.

The research problem addressed is the model-based forecasting of crisis effects accompanying transformations of biodiversity. During aggressive invasion processes — when a new species characterized by a high reproductive parameter r enters a biotic environment — several distinct scenarios of transitional dynamics may emerge. Obtaining biophysically realistic oscillatory dynamics remains problematic for the entire class of known equations. Continuous delay-differential equations $\dot{N} = rF(N(t - f(\tau)))$ exhibit a different biophysical deficiency inherent in their well-known [1] oscillatory solutions $N_*(t; r\tau)$. When large values of $r\tau$ are prescribed in an attempt to generate outbreak-type oscillations for an invasive species, the peaks of the solutions become sharply compressed in time while simultaneously stretched in amplitude; the intervening minima become excessively deep, prolonged and unrealistically low. Real populations cannot persist for long under such model trajectories. Consider Hutchinson cyclic equation key features for biophysics and epidemiological studies as a model of invasions following the results of [2–4] where the only dimensionless parameter:

$$\dot{N}(t) = rN(t)\big(1 - N(t-1)\big), \qquad r > 0. \tag{1}$$

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For every $r > \pi/2$ there exists a unique (up to time shifts) exponentially orbitally stable $T(r)$-periodic solution $N_*(t;r)$. Let $r_{\max} > \pi/2$ be a fixed, ecologically admissible but sufficiently large value of the r. We consider the compact interval $r \in \left[\pi/2+\delta, r_{\max}\right]$ for an arbitrary but fixed $\delta > 0$. Denote by $t^0_{\min}(r)$ the instant of the first minimum of $N_*(t;r)$ on a fundamental period. The complete sequence of minima is $t^k_{\min}(r) = t^0_{\min}(r) + k\,T(r), k \in \mathbb{Z}$. When $r_{\max}$ is sufficiently large the period admits the asymptotic representation $T(r) = \exp\bigl(r\bigl(1+\varphi(r)\bigr)\bigr)$, where φ is a continuous residual function satisfying $\varphi(r) \to 0 \quad$ as $\quad r \to +\infty$ (and therefore remains small on the whole interval up to $r_{\max}$). Because solutions of delay equations are in general asymmetric with respect to their extrema, the fractions of the period spent by the trajectory in a fixed ε-neighbourhood of 0 before and after each minimum are different. We introduce two continuous functions $\alpha_-, \alpha_+ : \left[\pi/2+\delta, r_{\max}\right] \to (0,1)$ and set $\delta_-(r) = \alpha_-(r)\,T(r), \delta_+(r) = \alpha_+(r)\,T(r)$. The set of all instants at which $N_*(t;r)$ remains inside the ε-neighbourhood of 0 is denoted by $\mathcal{I}_\varepsilon(r)$ and admits the representation: $\mathcal{I}_\varepsilon(r) = \bigl\{t \in \mathbb{R} : \ \bigl|N_*(t;r)\bigr| < \varepsilon\bigr\} = \bigcup_{k\in\mathbb{Z}}\Bigl(t^k_{\min}(r) - \alpha_-(r)\,T(r),\ t^k_{\min}(r) + \alpha_+(r)\,T(r)\Bigr)$. Substituting the asymptotic expression for the period yields the equivalent form $\mathcal{I}_\varepsilon(r) = \bigcup_{k\in\mathbb{Z}}\Bigl(t^k_{\min}(r) - \alpha_-(r)\exp\bigl(r\bigl(1+\varphi(r)\bigr)\bigr),\ t^k_{\min}(r) + \alpha_+(r)\exp\bigl(r\bigl(1+\varphi(r)\bigr)\bigr)\Bigr)$. As $r \to r_{\max}$ one has $T(r) \to T(r_{\max}) = \exp\bigl(r_{\max}\bigl(1+\varphi(r_{\max})\bigr)\bigr) < \infty, \alpha_\pm(r) \to \alpha_\pm(r_{\max})$, and, in the already well-developed relaxation regime, $\alpha_-(r_{\max}) + \alpha_+(r_{\max}) \approx 1$. Consequently the total length of each connected component of $\mathcal{I}_\varepsilon(r)$ tends to the finite (yet possibly very large) value $\bigl(\alpha_-(r_{\max})+\alpha_+(r_{\max})\bigr)\exp\bigl(r_{\max}\bigl(1+\varphi(r_{\max})\bigr)\bigr)$. Thus, when the $r-$parameter attains its maximal admissible value, the set $\mathcal{I}_\varepsilon(r)$ saturates: the intervals become as long as the ecological constraints permit, while remaining finite.

The aim of the study is to develop method based on hybrid automata for representing transforming oscillatory regimes of extreme biophysical processes, grounded in models of types of real situations (invasions, epidemics, chronic viral infections). We have substantiated in [5] that oscillatory regimes, even within a single invasive process, cannot be described by one generalized model. In situation of active resistance, which is formed against a rapidly multiplying population, has a regulation function $U(N(t-\tau))$ and F with another delay statement $F(N(t-\tau_1))\,\tau_1$ in form of counteraction equation:$\dot{N} = rN(t)U(N(t-\tau)) - F(N(t-\tau_1))$.

Let $N_t(\theta) = N(t+\theta)$, $\theta \in [-\max(\tau,\tau_1), 0]$, [6] denote the history segment of the population density. The following reproductive cycle-generating as

$rN(t)U(N_t)$ functions we will choose from aspects of situational conditions:

$$U_G(N_t) = \ln \frac{K}{N(t-\tau)} \qquad \text{(delayed Gompertz)}, \tag{2}$$

$$U_H(N_t) = \Big(1 - \frac{N(t-\tau)}{K}\Big) \qquad \text{(Hutchinson / delayed logistic)}, \tag{3}$$

$$U_R(N_t) = \exp\big(-\alpha N(t-\tau)\big) \qquad \text{(delayed Ricker)}. \tag{4}$$

The damping (cycle-suppressing) functionals that explicitly involve the critical level J are

$$F_S(N_t) = \frac{\mu N^m(t-\tau_1)}{(J-N(t))^2}, \qquad m \in [2,4], \quad \text{(singular harvesting)}, \tag{5}$$

$$F_M(N_t) = \frac{\mu N^m(t-\tau_1)}{J^2 - N^2(t)} \qquad \text{(mild rational damping)}, \tag{6}$$

$$F_E(N_t) = \mu N^m(t-\tau_1) \exp\Big(\frac{\gamma}{J-N(t)}\Big) \qquad \text{(exponentially damping)}. \tag{7}$$

To simulate the change in stages of invasive processes fading in the refugium, we will choose the change in population regimes as follows:

$$\frac{dN}{dt} = \begin{cases} N(t)\, r \ln \dfrac{K}{N(t-\tau)}, & N < \dfrac{J}{2}, \\ N(t)\, r \ln \dfrac{K}{N(t-\tau)} - \dfrac{\mu N^m(t-\tau_1)}{(J-N(t))^2}, & \dfrac{J}{2} \leqslant N < J, \\ rN(t)\Big(1 - \dfrac{N(t-\tau)}{K}\Big) - \dfrac{\mu N^m(t-\tau_1)}{J^2 - N^2(t)}, & N \geqslant J. \end{cases} \tag{8}$$

The resulting hybrid right-hand side is piecewise continuous on each open regime. Additional regulatory functions may tend to zero at the boundaries of their application ranges. Consequently, the initial-value problem is well-posed in the space of continuous histories, and solutions can cross the switching surfaces transversally under generic parameter conditions. For the model of waves of the local epidemic process of coronavirus in our method, it is necessary to select other combinations of generating and damping functions according to the described principle of constructing a hybrid structure.

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