The one-dimensional closest neighbor search problem solution using the cellular automata with locators [1]

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The paper considers applying the locator cellular automaton model to the closest neighbour search problem. The locator cellular automaton model assumes the possibility for each cell to translate a signal through any distance using ether. It is proven in this paper that such possibility allows to decrease the problem complexity from linear to logarithmic (against the classic cellular automaton model).

Keywords: cellular automaton, homogeneous structures,the closest neighbour search problem.

1. Introduction

Cellular automata (other names: self-reproducing automata and homogeneous structures) are discrete mathematical models of a wide class of real systems along with the processes taking place in them.

Theory of self-reproducing automaton was introduced by John von Neumann[2, 1] to describe the processes self-reproduction in biology and technology. His model was further developed and the term "Cellular automaton" as it described below was used by A. Burks [3], E. Moore [4], V. B. Kudryavtsev, A. S. Podkolzin, A. A. Bolotov [5] and other researchers.

Cellular automaton — is a mathematical object with discrete space and time. Its every position in space represented by a single cell, and each moment in time represented by discrete time step or generation. The state of each spatial cell is determined by very simple rules of interaction. These rules prescribe changes in the state of each cell in the next time step in response to the current state of neighboring cells.

In the paper of Gasanov E.E. [9] the concept of a cellular automaton with locators was introduced, which differs from the concept of a classic cellular automaton in that it allows the transmission of information not only between neighboring cells, but also at any distance, by means of transmitting a signal to the ether. The paper considers the application of this model to the one-dimensional closest neighbor search problem: a special point called

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"central"and some finite set of "target"cells are arbitrarily marked on $\mathbb{Z}^1$; the problem is to understand which of the target points is closer to the central one. The classic model of a cellular automaton solves this problem in linear time (by the minimal distance between the central and the target points). In this paper, it will be shown that the problem can be solved in logarithmic time via the cellular automaton with locators model.

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2. The problem description and results formulation.

In the paper of Gasanov E.E. [9] the concept of a cellular automaton with locators was introduced. Here we will give this concept, narrowing it down to the one-dimensional case.

By a *solid angle* in $\mathbb{R}^k$ we mean the union of all the rays in the space $\mathbb{R}^k$ emanating from a given point (*vertex of an angle*) and intersecting some hypersurface in $\mathbb{R}^k$. In the definition, we assume that a solid angle does not contain its vertex. In particular, in this paper we consider two degenerate cases: the full solid angle coinciding with $\mathbb{R}^k$ without the vertex of the angle, which we denote by Ω, and solid angles equal to one ray. If a solid angle is a ray, we denote it by a vector defining its direction.

A *cellular automaton with locators* is a 8-tuple

$$\sigma = (\mathbb{Z}^k, E_n, V, E_q, +, L, \varphi, \psi)$$

where $\mathbb{Z}^k$ is the set of k-dimensional vectors with integer coordinates, $E_n = \{0, 1, \ldots, n-1\}$, $V = (\alpha_1, \ldots, \alpha_{h-1})$ is an ordered set of pairwise different nonzero vectors from $\mathbb{Z}^k$, $E_q = \{0, 1, \ldots, q-1\}$, $+$ is a commutative semigroup operation defined on E_q, $L = (\nu_1, \ldots, \nu_m)$ is an ordered set of pairwise different solid angles in $\mathbb{R}^k$ with a vertex at the origin, $\varphi : E_n^h \times E_q^m \to E_n$ is a function depending on the variables $x_0, x_1, \ldots, x_{h-1}, z_1, \ldots, z_m$ such that $\varphi(0, \ldots, 0) = 0$, $\psi : E_n^h \times E_q^m \to E_q$ is a function that depends on the variables $x_0, x_1, \ldots, x_{h-1}, z_1, \ldots, z_m$. Here the variables $x_0, x_1, \ldots, x_{h-1}$ take values from E_n and the variables $z_1, \ldots, z_m$ take values from E_q. Elements of the set $\mathbb{Z}^k$ are called *cells* of the cellular automaton σ; elements of the set E_n are called *cell states* of the cellular automaton σ; the set V is called the *neighborhood pattern* of the cellular automaton σ; elements of the set E_q are called *broadcasting signals*; the set L is called the *locator pattern* of the cellular automaton σ; the function φ is called the *local transition function* of the automaton σ; the function ψ is called the *broadcasting function* of the automaton σ. The state 0 is interpreted as *rest state* and the condition $\varphi(0, \ldots, 0) = 0$ is interpreted as a condition for maintaining the rest state.

Here we need to introduce an ordering of the neighborhood pattern V and the locator pattern L in order to establish a one-to-one correspondence between vectors from V and solid angles from L and variables $x_0,\ x_1, \ldots, x_{h-1},\ z_1, \ldots, z_m$ of the local transition function φ and the broadcasting function ψ respectively. We can make this correspondence more explicit if we index the variables of the functions φ and ψ by the vectors and solid angles themselves, i.e. assume that the local transition function φ and the broadcasting function ψ depend on the variables $x_{\mathbf{0}}$, $x_{\alpha_1}, \ldots, x_{\alpha_{h-1}}, z_{\nu_1}, \ldots, z_{\nu_m}$, where the index of the first variable is the zero vector $\mathbf{0} = (0, \ldots, 0) \in \mathbb{Z}^k$. If we index the variables of the local transition function and broadcasting function in this way, we can write them in any order, and then we can define the neighborhood pattern and the locator pattern simply as a set, not an ordered set.

In the rest of this section we use these conventions: consider the neighborhood pattern as a set of vectors, and the locator pattern as a set of solid angles and index the variables of the local transition function and broadcasting function by the vectors from the neighborhood pattern and solid angles from the locator pattern. At the same time, we often omit the outer parentheses of the vectors in the indices. For example, if $k = 2$, $n = 2$, $q = 2$, $V = \{(-1, 0), (1, 0)\}$, and $L = \{\Omega, (0, 1)\}$, then a local transition function may look like this: $\varphi = x_{-1,0} \& z_\Omega \vee x_{1,0} \& z_{0,1}$.

If $\alpha \in \mathbb{Z}^k$, ν is a solid angle with vertex at the origin, then by $\nu(\alpha)$ we denote the solid angle obtained by translation of the angle ν to the point α.

If $\alpha \in \mathbb{Z}^k$ is a cell of a cellular automaton with locators σ, then the set $V(\alpha) = \{\alpha, \alpha + \alpha_1, \ldots, \alpha + \alpha_{h-1}\}$ is called the *neighborhood of the cell* α, and elements of the set $L(\alpha) = \{\nu_1(\alpha), \ldots, \nu_m(\alpha_m)\}$ are called *locators of the cell* α.

A *state of a cellular automaton with locators* σ is a pair (e, f), where e is an arbitrary function from the set $\mathbb{Z}^k$ to the set E_q, called *broadcast state*, f is an arbitrary function from the set $\mathbb{Z}^k$ to the set E_n and called *distribution of states of the cellular automaton with locators* σ. Such a function can be interpreted as a certain mosaic arising in the k-dimensional space as a result of assigning a certain state from the set E_n and some signal from the set E_q to each point with integer coordinates. The set of all possible states of a cellular automaton with locators is denoted by Σ.

If $\alpha \in \mathbb{Z}^k$ and (e, f) is a state of a cellular automaton with locators σ, then the value $e(\alpha)$ is called *the signal of the cell* α*, defined by the state* (e, f), and the value $f(\alpha)$ is *the state of the cell* α*, determined by the state* (e, f). For each $i \in \{1, \ldots, m\}$ the value

$$s_i(\alpha) = \sum_{\beta \in \nu_i(\alpha) \cap \mathbb{Z}^k} e(\beta) \tag{1}$$

we call *the value of the locator ν_i, determined by the state* (e, f). Here, in the summation the semigroup operation + defined on E_q is used.

On the set Σ we define the *global transition function* Φ of a cellular automaton with locators σ, putting $\Phi(e, f) = (e', f')$, where $(e, f), (e', f') \in \Sigma$ and for any cell $\alpha \in \mathbb{Z}^k$ the following identities hold

$$f'(\alpha) = \varphi(f(\alpha), f(\alpha + \alpha_1), \ldots, f(\alpha + \alpha_{h-1}), s_1(\alpha), \ldots, s_m(\alpha)), \quad (2)$$

$$e'(\alpha) = \psi(f(\alpha), f(\alpha + \alpha_1), \ldots, f(\alpha + \alpha_{h-1}), s_1(\alpha), \ldots, s_m(\alpha)). \quad (3)$$

A meaningful interpretation of the mapping Φ is that the signal of each cell and the state of each cell "after the transition" is determined by the state of the neighborhood of the cell and by the values of the locators "before the transition" using the rules ψ and φ in the same way for all cells.

By the *behavior of a cellular automaton with locators* σ we call a sequence $(e_0, f_0), (e_1, f_1), (e_2, f_2), \ldots$ of states such that the equation $(e_{i+1}, f_{i+1}) = \Phi(e_i, f_i)$ holds for all $i = 0, 1, 2, \ldots$. The state (e_i, f_i) is called the *state of the cellular automaton with locators σ at the time i*, and (e_0, f_0) is also called the *initial state of the cellular automaton with locators σ*.

Let's formulate the closest neighbour search problem on the line. Let the I be the initial state of a cellular automaton on $\mathbb{Z}^1$ which satisfies the following conditions:

1) Any cell is on one of $\{q_S; q_{C_0}, *\}$ states.

2) There is only one q_{C_0} cell.

3) There is a finite and non-empty set of q_S cells.

We will define that a cellular automaton state I' is solution for the problem I if I' satisfies the following conditions:

1) The q_{C_0} cell from I is in q_{CF} state in I'.

2) The cell which is the closest to the q_{C_0} cell in I is in the q_{LE} state if it's to the left and in q_{RE} state if it is ti the right. If there are two closest cell then the right cell must be in $*$ state and the left one — in q_{LE} state.

3) The cells which lie between q_{CF} and q_{LE} cells are in q_{LF} state. The cells which lie between q_{CF} and q_{RE} cells are in q_{RF} state.

4) The rest cells are in $*$ state.

We define that cellular automaton σ solves the closest neighbour search problem if it satisfies the following conditions:

1) If the initial state I of the cellular automaton is a closest neighbour search problem then the automaton must end up in I' state which is solution for I.

2) If the automaton takes state S which is solution for some closest neighbour search problem this state must be kept for all the next tacts.

There is a cell automaton with locators σ with 25 states and the ether alphabet power 12 which solves the closest neighbour search problem for not longer than $\log_2 s + 7$, where s is the distance between the q_{C_0} cell and the closest to it q_S cell.

No cell automaton with locators σ can solve the closest neighbour search problem faster than $\log_M(\frac{s}{5})$, where s is the distance between the q_{C_0} cell and the closest to it q_S cell and M is the ether alphabet power.

3. Formal automaton description

Let's consider cellular automaton $\sigma = (\mathbb{Z}^1, E_n, V, E_q, +, L, \varphi, \psi)$, where $V = \{(1), (-1)\}$, $E_q = \{0,1\}^2 \times \{0,1,2\}$, and $L = (\nu_{-1}, \nu_1)$, where ν_{-1}, ν_1 — degenerate solid angles, corresponding to vectors (-1) and (1).

Let's define the semigroup operation on E_q as follows: $(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, max(b_1, b_2), max(c_1, c_2))$

Let the state set $E_n = \{q^{=}_{CC}; q^{<}_{CC}; q^{>}_{CC}; q_S; q_{C_0}; q_L; q_R;$
$q^C_2; q^C_1; q_{C_{L_1}}; q_{C_{R_1}}; q_{LF}; q_{RF}; q_{LE}; q_{RE}; q_{CF}; q^L_*; q^R_*; q^L_1; q^R_1; q^L_0; q^R_0; q^*_L; q^*_R; *\}$

Let's define state $*$ as the rest state. The automaton is designed in a way that only a limited set of cells will be in a non-rest state on each tact. Considering that operation $+$ has the property $(0,0,0) + (0,0,0) = (0,0,0)$, we can conclude that any locator's value is calculated from a limited set of non-zero terms, so it is defined correctly.

Let's describe φ and ψ functions for each automaton state:

q^L_k and q^R_k, $k \in 0;1$ are the key autmaton states. If there are some amount of q^L_1 consecutive cells then on the next tact every second q^L_1 cell will go to the q^L_0 state and the rest q^L_1 cells will keep their state. It will be proven later that such behavior allows to translate a segment length bit by bit.

$$\varphi(q^L_k, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} q^L_{(k \wedge z^1_{\nu_{-1}})}, \text{ if } z^3_{\nu_1} = 0 \\ q_{LF}, \text{ if } z^3_{\nu_1} = 1 \\ * \text{ in other cases} \end{cases}, \tag{4}$$

$$\psi(q^L_k, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (k \wedge z^1_{\nu_{-1}}, k \wedge z^1_{\nu_{-1}}, 0), \tag{5}$$

$$\varphi(q_k^R, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} q^R_{(k \wedge z^1_{\nu_1})}, \text{ if }, z^3_{\nu_{-1}} = 0 \\ q_{RF}, \text{ if } z^3_{\nu_{-1}} = 2 \\ * \text{ in other cases} \end{cases}, \tag{6}$$

$$\psi^1(q_k^R, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (k \wedge z^1_{\nu_1}, k \wedge z^1_{\nu_1}, 0). \tag{7}$$

q_{C_0} is the initial central cell state. The cell in this state will translate $(0, 0, 1)$ signal for other cells to determine if they are on the left or on the right side.

$$\varphi(q_{C_0}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = q_1^C, \tag{8}$$

$$\psi(q_{C_0}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0, 0, 1). \tag{9}$$

q_S is the initial target cells state. The cell in this state will wait for a special signal to change it's state to the left or right version.

$$\varphi(q_S, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} q_L^*, \text{ if } z^3_{\nu_1} = 1 \\ q_R^*, \text{ if } z^3_{\nu_{-1}} = 1 \\ q_S \text{ in other cases} \end{cases}, \tag{10}$$

$$\psi(q_S, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} (0, 0, 1) \text{ if } z^3_{\nu_1} = 1 \\ (0, 1, 0) \text{ in other cases} \end{cases}. \tag{11}$$

$*$ is the initial state of internal cells (cells which are not central or target). If $*$-cell is part of the problem (i.e. it lies between the central cell and the side-closest target cell) it will wait for a special signal to change it's state on the left or right version. Otherwise such cell will keep calm as a rest cell.

$$\varphi(*, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} q_*^L, \text{ if } z^3_{\nu_1} = 1, z^2_{\nu_{-1}} = 1 \\ q_*^R, \text{ if } z^2_{\nu_1} = 1, z^3_{\nu_{-1}} = 1 \\ * \text{ in other cases} \end{cases}, \tag{12}$$

$$\psi(*, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0, 0, 0). \tag{13}$$

States q_*^L and q_*^R are designed for two purposes. First of all, they will provide right ether structure right before the main part of the algorithm will begin. Secondly, internal cells in this state are waiting for a special signal to determine if they lie between the side-closest and the central cell. If they are not, they go to the rest state.

$$\varphi(q_*^L, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} * \text{ if } z_{\nu_1}^3 = 1 \\ q_1^L \text{ in other cases} \end{cases}, \tag{14}$$

$$\psi(q_*^L, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} (0,0,0) \text{ if } z_{\nu_1}^3 = 1 \\ (0,1,1) \text{ in other cases} \end{cases}, \tag{15}$$

$$\varphi(q_*^R, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} * \text{ if } z_{\nu_{-1}}^2 = 1 \\ q_1^R \text{ in other cases} \end{cases}, \tag{16}$$

$$\psi(q_*^R, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} (0,0,0) \text{ if } z_{\nu_{-1}}^2 = 1 \\ (0,1,1) \text{ in other cases} \end{cases}. \tag{17}$$

q_L^* and q_R^* are special target cells states. A cell in this state will wait for a special signal to determine if it is the side-closest target cell. If it is not, the cell go to the rest state.

$$\varphi(q_L^*, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} * \text{ if } z_{\nu_1}^3 = 1 \\ q_L \text{ in other cases} \end{cases}, \tag{18}$$

$$\psi(q_*^L, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0,0,0), \tag{19}$$

$$\varphi(q_R^*, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} * \text{ if } z_{\nu_{-1}}^2 = 1 \\ q_R \text{ in other cases} \end{cases}, \tag{20}$$

$$\psi(q_*^R, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0,0,0). \tag{21}$$

$q_{CC}^{=}, q_{CC}^{>}, q_{CC}^{<}$ are working central cell states. The central cell compares lengths of the left and the right segments. Automaton works in a way that those lengths binary notation come to the central cell as ether signals: from the lowest bit to the highest. The central cell can change it's status depending on the current bit pair: $q_{CC}^{>}$ — if current left bit is greater than the right one, $q_{CC}^{<}$ — if it is less. If the left bit is equal to the right one, the central cell state inherits from the previous tact. State $q_{CC}^{=}$ occurs in the beginning and may stay until the first non-equal bit pair.

$$\varphi(q_{CC}^X, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} q_{CC}^X, \text{ if } z_{\nu_{-1}}^2 = 1, z_{\nu_1}^2 = 1, z_{\nu_{-1}}^1 = z_{\nu_1}^1 \\ q_{CC}^{>}, \text{ if } z_{\nu_{-1}}^2 = 1, z_{\nu_1}^2 = 1, z_{\nu_{-1}}^1 > z_{\nu_1}^1 \\ q_{CC}^{<}, \text{ if } z_{\nu_{-1}}^2 = 1, z_{\nu_1}^2 = 1, z_{\nu_{-1}}^1 < z_{\nu_1}^1 \\ q_{CF} \text{ in other cases} \end{cases}, \tag{22}$$

$$\psi(q_{C_1}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} (0,0,0), \text{ if } z^2_{\nu_{-1}} = 1, z^2_{\nu_1} = 1 \\ (0,0,1), \text{ if } (z^2_{\nu_{-1}} = 0 \vee z^2_{\nu_1} = 0) \wedge \\ \wedge((z^2_{\nu_{-1}} > z^2_{\nu_1}) \vee (z^2_{\nu_{-1}} = z^2_{\nu_1} \wedge q^X_{CC} \neq q^{>}_{CC})) \\ (0,0,2) \text{ in other cases} \end{cases}. \tag{23}$$

q_L and q_R are side-closest target cells states. They just mark the end of the segment and go to their final state when hear a special ether signal.

$$\varphi(q_L, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} q_{LE}, \text{ if } z^3_{\nu_1} = 1 \\ *, \text{ if } z^3_{\nu_1} = 2 \\ q_L \text{ in other cases} \end{cases}, \tag{24}$$

$$\psi(q_L, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0,0,0), \tag{25}$$

$$\varphi(q_R, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} q_{RE}, \text{ if } z^3_{\nu_{-1}} = 2 \\ *, \text{ if } z^3_{\nu_{-1}} = 1 \\ q_R \text{ in other cases} \end{cases}, \tag{26}$$

$$\psi(q_R, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0,0,0). \tag{27}$$

q^C_2 state is designed to for the central cell to looks for a special signal from each side so it could find out if there is at least one target cell at each side. If not, the problem is much easier and can be resolved in a constant time.

$$\varphi(q^C_2, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \begin{cases} q_{C_{L_1}}, \text{ if } z^2_{\nu_1} = 0 \\ q_{C_{R_1}}, \text{ if } z^2_{\nu_{-1}} = 0 \\ q^C_1, \text{ in other cases} \end{cases}, \tag{28}$$

$$\psi(q^C_2, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0,0,0). \tag{29}$$

q^C_1 is a 1-tact sleep state. The central cell in this state don't send anything in the ether.

$$\varphi(q^C_1, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = q_{C_1}, \tag{30}$$

$$\psi(q^C_1, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0,0,0). \tag{31}$$

$q_{C_{L_1}}$ and $q_{C_{R_1}}$ are states which occur when the problem is trivial in a way that all the target cells are located at the same side of the central cell.

$$\varphi(q_{C_{L_1}}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = q_{CF}, \tag{32}$$

$$\psi(q_{C_{L_1}}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0, 0, 1), \tag{33}$$

$$\varphi(q_{C_{R_1}}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = q_{CF}, \tag{34}$$

$$\psi(q_{C_{R_1}}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0, 0, 2). \tag{35}$$

ψ and φ functions below are function for the finish states. Those state do not change or send anything to the ether.

$$\varphi(q_{LF}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = q_{LF}, \tag{36}$$

$$\psi(q_{LF}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0, 0, 0), \tag{37}$$

$$\varphi(q_{RF}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = q_{RF}, \tag{38}$$

$$\psi(q_{RF}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0, 0, 0), \tag{39}$$

$$\varphi(q_{LE}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = q_{LE}, \tag{40}$$

$$\psi(q_{LE}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0, 0, 0), \tag{41}$$

$$\varphi(q_{RE}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = q_{RE}, \tag{42}$$

$$\psi(q_{RE}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0, 0, 0), \tag{43}$$

$$\varphi(q_{CF}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = q_{CF}, \tag{44}$$

$$\psi(q_{CF}, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = (0, 0, 0). \tag{45}$$

Let's define *left cells* as cells with L character in the state name (and correspondingly define *right cells*). Let $Q_1 = \{q_k^X\}$, where $X \in \{R, L\}, k = 1$, $Q_0 = \{q_k^X\}$, where $X \in \{R, L\}, k = 0$.

4. Automaton's behavior.

Let's describe automaton's behavior on each algorithm phase. We also will provide a simple example for better understanding. We will describe automaton states as follows:

	t=0															
Q	q_S	$*$	q_S	$*$	$*$	$*$	$*$	$*$	$*$	q_{C_0}	$*$	$*$	$*$	$*$	$*$	q_S
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
ψ^3	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0

Where $Q = q_1, q_2, ..., q_n$ — automaton cells states. We only consider cells which lie between non-$*$ cell in the initial automaton state. All q_S and q_{C_0} cells are also considered.

$L^1 = l_1^1, l_2^1, ..., l_n^1$ — first components of the left ether sum.

$L^2 = l_1^2, l_2^2, ..., l_n^2$ — second components of the left ether sum.

$L^3 = l_1^3, l_2^3, ..., l_n^3$ — third components of the left ether sum.

$R^1 = r_1^1, r_2^1, ..., r_n^1$ — first components of the right ether sum.

$R^2 = r_1^2, r_2^2, ..., r_n^2$ — second components of the right ether sum.

$R^3 = r_1^3, r_2^3, ..., r_n^3$ — third components of the right ether sum.

$\psi^1 = \psi_1^1, \psi_2^1, ..., \psi_n^1$ — the first component of the signal, being sent to the ether

$\psi^2 = \psi_1^2, \psi_2^2, ..., \psi_n^2$ — the second component of the signal, being sent to the ether

$\psi^3 = \psi_1^3, \psi_2^3, ..., \psi_n^3$ — the third component of the signal, being sent to the ether

4.1. Phase 1: determine the orientation.

Let the initial automaton state satisfy the Theorem 1 conditions:

	t=0															
Q	q_S	$*$	q_S	$*$	$*$	$*$	$*$	$*$	$*$	q_{C_0}	$*$	$*$	$*$	$*$	$*$	q_S
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
ψ^3	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0

There are only $(0, 0, 0)$ signals in the ether on the first tact. Transition functions of q_S and $*$ states are equal to q_S and $*$ respectively when

the input ether sums are $(0,0,0)$. Broadcasting functions of these states are equal to $(0,1,0)$ and $(0,0,0)$ respectively. The transition function of the q_{C_0} state is equal to q_2^C, and the broadcasting function takes the value $(0,0,1)$. Overall, only the central cell will change it's state on the first tact. Besides, the ether space will fill with central and target cells signals:

	t=1															
Q	q_S	$*$	q_S	$*$	$*$	$*$	$*$	$*$	$*$	q_2^C	$*$	$*$	$*$	$*$	$*$	q_S
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
L^3	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
R^3	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
ψ^3	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0

The transition functions values of q_S and $*$ states depend on the side from which signals $(0,1,0)$ and $(0,0,1)$ came. q_S cells could either go to the q_L^* state and translate a $(0,0,1)$ signal or go to the q_R^* state and translate a $(0,1,0)$ signal. $*$ cells can go to q_*^L or q_*^R states and always output a $(0,0,0)$ signal. q_2^C cell looks for a $(0,1,0)$ signal from each side. If such a signal is only present on one side, central cell goes to $q_{C_{L_1}}$ or $q_{C_{R_1}}$ state (depending on the side on which $(0,1,0)$ signal occurs) and we don't need to compare segments from different sides. This is an easy case and the problem can be resolved in a constant time at this point. In our case the signal is present on both sides so the central cell goes to the q_2^C state to sleep for 2 tacts and wait before the other cells are ready to begin the length calculation:

	t=2															
Q	q_L^*	q_*^L	q_L^*	q_*^L	q_*^L	q_*^L	q_*^L	q_*^L	q_*^L	q_1^C	q_*^R	q_*^R	q_*^R	q_*^R	q_*^R	q_R^*
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^3	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
R^3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	1	1	1	1	1	1	0	1	1	1	1	1	0
ψ^2	0	0	0	1	1	1	1	1	1	0	1	1	1	1	1	0
ψ^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

This is how Phase 1 ends. To sum up, from all the initial q_S cells we only keep the two closest to the center. Those cells are now oriented (i.e. are either in q_L or q_R states); the $*$ cells which lie between the central cell and one of the two kept target cells will also go to the oriented state q_*^L or q_*^R; the central cell will go to the q_1^C state.

4.2. Phase 2: length comparison.

This is a complicated iterative phase.

We will call q_L and q_R cells *extreme*, and cells which lie between the central and an extreme cell — *internal*. Let's define that this phase

ends when the central cell translate a signal $\psi \in \{(0,0,1);(0,0,2)\}$. The transition functions are designed in a way that the central cell is always in $q^{=}_{CC}, q^{>}_{CC}, q^{<}_{CC}$ state in Phase 2 and it can only go to another state when it send one of $\{(0,0,1);(0,0,2)\}$ signals. Finally, it's easy to see that signals $\{(0,0,1);(0,0,2)\}$ are only sent when all the internal cells from some side are in $q \in Q_0$ state.

Let's consider the left side of the cellular automaton. We will enumerate the cells from left to right and denote $q^{i,t}, i \in \{1,2,..s_L\}, t \geq 0$ as an i-th cell state in the t-th tact. We assume that $t = 0$ — the beginning of Phase 2.

$q^{i,t} \in Q_1$ if and only if $i(mod\ 2^t) = 0$.

Доказательство. Let 's prove this statement by induction:

At the $t = 0$ moment all the cells are $q^L_1 \in Q_1$ states, which is equivalent to $i(mod\ 2^0) = 0$.

Let's prove the Lemma for $t = 1$. From the transition functions of the $\{q^L_1; q^R_1; q^L_0; q^R_0\}$ states we can see that $q^{i,1} \in Q_1 \leftrightarrow q^{i,0} \in Q_1 \wedge z^{i,0}_{\nu_{-1}} = (1,\alpha,\beta) \leftrightarrow z^{i,0}_{\nu_{-1}} = (1,\alpha,\beta)$ (here and further α and β are arbitrary elements of the {0;1} and {0;1;2} sets respectively). We know that $z^{i,0}_{\nu_{-1}} = \sum_{j=1}^{i} \psi(q^L_*, q_{-1}, q_1, z_{\nu_{-1}}, z_{\nu_1}) = \sum_{j=1}^{i-1}(1,1,0) = (i(mod\ 2),1,0)$, therefore $q^{i,1} \in Q_1 \leftrightarrow i(mod\ 2) = 0$.

Let's assume the Lemma is true for $t = 0,1,2,...,k$ and prove it is true for $t = k+1$. From the transition functions of the $\{q^L_1; q^R_1; q^L_0; q^R_0\}$ states we can see that $q^{i,k+1} \in Q_1 \leftrightarrow q^{i,k} \in Q_1 \wedge z^{i,k}_{\nu_{-1}} = (1,\alpha,\beta)$. The broadcasting functions of the $\{q^L_1; q^R_1; q^L_0; q^R_0\}$ states are designed in a way that $\psi^{i,k-1} = (1,\alpha,\beta) \leftrightarrow q^{i,k} \in Q_1$, therefore $z^{i,k}_{\nu_{-1}} = (\sum_{j(mod\,2^k)=0, j<i, j>0} 1, \alpha, \beta)$. Thus, by the induction assumption:

$q^{i,k+1} \in Q_1 \leftrightarrow q^{i,k} \in Q_1 \wedge z^{i,k}_{\nu_{-1}} = (1,\alpha,\beta) \leftrightarrow i(mod\ 2^k) = 0 \wedge \sum_{j(mod\,2^k)=0, j<i, j>0} 1 = 1 \leftrightarrow i(mod\ 2^k) = 0 \wedge (i(mod\ 2^{k+1}) > 2^k \vee i(mod\ 2^{k+1}) = 0) \leftrightarrow i(mod\ 2^{k+1}) = 0$ □

We can also prove such lemma for the right side because the transition and broadcasting functions of the left and right cells only depend on left and right ether signals respectively.

It follows from the Lemma that at the t moment the central cell hear the value $\sum_{i(mod\,2^t)=0, i\in[1;s]} 1 = \lfloor \frac{s}{2^t} \rfloor (mod\,2)$ at the 1-st ether dimension, which is equal to $(t+1)$-th bit in s length binary notion. Thus, the central cell can compare the values s_L and s_R bit by bit, from the lowest to the highest position. The transition and broadcasting functions of the q^X_{CC}, $X \in \{=, <, >\}$ states implement such length comparison algorithm: if left i-th bit is

greater than such bit from the right then the central cell goes to the $q^{>}_{CC}$ state; if it is lower — to the $q^{<}_{CC}$ state; if they are equal, the state doesn't change. This algorithm continue working until one of the lengths translation ends ($z^2_{\nu_{-1}} = 0 \vee z^2_{\nu_1} = 0$). In this moment the central cell sends one of the signals $(0,0,2)$; $(0,0,1)$ depending on which side appeared to be shorter.

Phase 2 process for the given example:

	t=3															
Q	$*$	$*$	q_L	q^L_1	q^L_1	q^L_1	q^L_1	q^L_1	q^L_1	$q^{=}_{CC}$	q^R_1	q^R_1	q^R_1	q^R_1	q^R_1	q_R
L^1	0	0	0	0	1	0	1	0	1	0	0	1	0	1	0	1
L^2	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	1	1	1	0	1	0	1	0	1	1	0	1	0	1	0	0
R^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	1	0	1	0	1	0	0	1	0	1	0	0
ψ^2	0	0	0	0	1	0	1	0	1	0	0	1	0	1	0	0
ψ^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	t=4															
Q	$*$	$*$	q_L	q^L_0	q^L_1	q^L_0	q^L_1	q^L_0	q^L_1	$q^{<}_{CC}$	q^R_0	q^R_1	q^R_0	q^R_1	q^R_0	q_R
L^1	0	0	0	0	0	1	1	0	0	1	1	1	0	0	1	1
L^2	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	1	1	1	1	0	0	1	1	0	0	0	1	1	0	0	0
R^2	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
ψ^2	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
ψ^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	t=5															
Q	$*$	$*$	q_L	q^L_0	q^L_0	q^L_0	q^L_1	q^L_0	q^L_0	$q^{>}_{CC}$	q^R_0	q^R_1	q^R_0	q^R_0	q^R_0	q_R
L^1	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0
L^2	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0
R^2	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	t=6															
Q	$*$	$*$	q_L	q^L_0	q^L_0	q^L_0	q^L_0	q^L_0	q^L_0	$q^{>}_{CC}$	q^R_0	q^R_0	q^R_0	q^R_0	q^R_0	q_R
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^3	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0

4.3. Phase 3: finalize.

This is a very straight forward phase. It begins after the central cell send on of the signals $\{(0,0,1),(0,0,2)\}$ during Phase 2. Let's assume that the left side were longer and it was the $(0,0,2)$ signal:

t=6

Q	$*$	$*$	q_L	q_0^L	q_0^L	q_0^L	q_0^L	q_0^L	q_0^L	$q_{CC}^{>}$	q_0^R	q_0^R	q_0^R	q_0^R	q_0^R	q_R
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^3	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0

This signal will take place in the ether in the next tact:

t=7

Q	$*$	$*$	q_L	q_0^L	q_0^L	q_0^L	q_0^L	q_0^L	q_0^L	q_{CF}	q_0^R	q_0^R	q_0^R	q_0^R	q_0^R	q_R
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^3	0	0	0	0	0	0	0	0	0	0	2	2	2	2	2	2
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^3	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

When left cells hear this signal they go to the $*$ state. The right target cell goes to the q_{RE} state and the right internal cells — to the q_{RF} state:

t=8

Q	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	q_{CF}	q_{RF}	q_{RF}	q_{RF}	q_{RF}	q_{RF}	q_{RE}
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

5. Complete calculation process for the given example

For clarity we write the whole process for the given example below:

t=0

Q	q_S	$*$	q_S	$*$	$*$	$*$	$*$	$*$	$*$	q_{C_0}	$*$	$*$	$*$	$*$	$*$	q_S
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
ψ^3	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0

t=1

Q	q_S	$*$	q_S	$*$	$*$	$*$	$*$	$*$	$*$	q_2^C	$*$	$*$	$*$	$*$	$*$	q_S
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
L^3	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
R^3	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
ψ^3	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0

t=2

Q	q_L^*	q_*^L	q_L^*	q_*^L	q_*^L	q_*^L	q_*^L	q_*^L	q_*^L	q_1^C	q_*^R	q_*^R	q_*^R	q_*^R	q_*^R	q_R^*
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^3	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
R^3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	1	1	1	1	1	1	0	1	1	1	1	1	0
ψ^2	0	0	0	1	1	1	1	1	1	0	1	1	1	1	1	0
ψ^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

t=3

Q	$*$	$*$	q_L	q_1^L	q_1^L	q_1^L	q_1^L	q_1^L	q_1^L	$q_{CC}^=$	q_1^R	q_1^R	q_1^R	q_1^R	q_1^R	q_R
L^1	0	0	0	0	1	0	1	0	1	0	0	1	0	1	0	1
L^2	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	1	1	1	0	1	0	1	0	1	1	0	1	0	1	0	0
R^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	1	0	1	0	1	0	0	1	0	1	0	0
ψ^2	0	0	0	0	1	0	1	0	1	0	0	1	0	1	0	0
ψ^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

t=4

Q	$*$	$*$	q_L	q_0^L	q_1^L	q_0^L	q_1^L	q_0^L	q_1^L	$q_{CC}^<$	q_0^R	q_1^R	q_0^R	q_1^R	q_0^R	q_R
L^1	0	0	0	0	0	1	1	0	0	1	1	1	0	0	1	1
L^2	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	1	1	1	1	0	0	1	1	0	0	0	1	1	0	0	0
R^2	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
ψ^2	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
ψ^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	t=5															
Q	$*$	$*$	q_L	q_0^L	q_0^L	q_0^L	q_1^L	q_0^L	q_0^L	$q_{CC}^>$	q_0^R	q_1^R	q_0^R	q_0^R	q_0^R	q_R
L^1	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0
L^2	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0
R^2	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	t=6															
Q	$*$	$*$	q_L	q_0^L	q_0^L	q_0^L	q_0^L	q_0^L	q_0^L	$q_{CC}^>$	q_0^R	q_0^R	q_0^R	q_0^R	q_0^R	q_R
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^3	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0

	t=7															
Q	$*$	$*$	q_L	q_0^L	q_0^L	q_0^L	q_0^L	q_0^L	q_0^L	q_{CF}	q_0^R	q_0^R	q_0^R	q_0^R	q_0^R	q_R
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^3	0	0	0	0	0	0	0	0	0	0	2	2	2	2	2	2
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^3	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	t=8														
Q	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	q_{CF}	q_{RF}	q_{RF}	q_{RF}	q_{RF}	q_{RF}	q_{RE}
L^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
R^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ψ^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

6. Automaton runtime

Let's calculate the time T need for σ to work out. From the previous section:

$$T = T_1 + T_2 + T_3. \tag{46}$$

where T_k is duration of corresponding phase. It's easy to see that $T_1 = 3$, $T_3 = 2$. It follows from the Lemma 2 that if Phase 2 runtime is t then $t = \min\limits_{2^r > s} r = \lceil \log_2(s + 0.5) \rceil$, where $s = \min(s_L, s_R)$. Because we start time t from 0 in Lemma 1, the overall runtime of Phase 2 will be:

$$T_2 = \lceil \log_2(s + 0.5) \rceil + 1 \leq \log_2(s) + 2.$$

Therefore,

$$T \leq \log_2(s) + 7.$$

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