Towards Relevant Multilatticce Logic

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The paper introduces the concept of de Morgan multimonoid and considers the possibility of constructing a relevant multilattice logic on its basis. The problem of constructing a sequent calculus for it is discussed.

Keywords: relevant logic, multilattice logic, multilattice, multimonoid, sequent calculus.

Multilattice logic $\mathbf{ML}_n$ arose in [4] as a generalisation of bilattice logic, trilattice logic, and tetralattice logic which are themselves generalisations of the logic of De Morgan lattices called first degree entailment, one of the most important relevant logics (see [4] for more details and [1] for the current state of affairs in multilattice logic). The semantics of $\mathbf{ML}_n$ is based on multilattices, i.e. lattices equipped with n partial orders. Meet, join, and inversion exist for any for any such order. Thus, a multilattice has n conjunctions, disjunctions, and negations. In [3] $\mathbf{ML}_n$ was supplied with n implications and co-implications. However, the implications added to $\mathbf{ML}_n$ are classical. Besides, the combination of two negations produced by different orders behaves as a Boolean complementation. Hence, $\mathbf{ML}_n$ is not a relevant logic, although its roots go back to relevant logic. We think that it is important to return multilattice logic to its motherland, relevant logic. As a solution of this problem, we suggest a combination of $\mathbf{ML}_n$ and the relevant logic $\mathbf{R}$. [4] A *multilattice* is a structure $\mathfrak{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$, where $n > 1$, $S \neq \emptyset$, $\leqslant_1, \ldots, \leqslant_n$ are partial orders such that $\langle S, \leqslant_1 \rangle, \ldots, \langle S, \leqslant_n \rangle$ are lattices with the corresponding pairs of meet and join operations $\langle \cap_1, \cup_1 \rangle$, $\ldots$, $\langle \cap_n, \cup_n \rangle$ and the corresponding j-inversion operations $\sim_1, \ldots, \sim_n$ which satisfy the following conditions, for each $j, k \leqslant n$, $j \neq k$, and $a, b \in S$:

$$a \leqslant_j b \text{ implies } \sim_j b \leqslant_j \sim_j a; \tag{anti}$$

$$a \leqslant_k b \text{ implies } \sim_j a \leqslant_k \sim_j b; \tag{iso}$$

$$\sim_j \sim_j a = a. \tag{per2}$$

An Abelian multimonoid is a structure $\mathfrak{A}_n = \langle S, \circ_1, \ldots, \circ_n, 1 \rangle$, where $n > 1$, $S \neq \emptyset$, each $\circ_j$ ($j \leqslant n$) is commutative, associative binary operation on S with 1 its identity, i.e. $1 \in S$ and $1 \circ_j a = a$, for each $a \in S$. A de Morgan multimonoid is a structure $\mathfrak{D}_n = \langle D, \leqslant_1, \ldots, \leqslant_n, \circ_1, \ldots, \circ_n, 1 \rangle$, where $n > 1$ and

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1) $\langle D, \leqslant_1, \ldots, \leqslant_n \rangle$ is a multilattice;

2) $\langle D, \circ_1, \ldots, \circ_n, 1 \rangle$ is an Abelian multimonoid;

3) the multimonoid is ordered by the multilattice, i.e. $a \circ_j (b \cup_t c) = (a \circ_j b) \cup_t (a \circ_j c)$, for each $j, t \leqslant n$;

4) $\circ_j$ is upper semi-idempotent ('square increasing'), i.e. $a \leqslant_t a \circ_j a$, for each $j, t \leqslant n$;

5) for each $j, t \leqslant n$, $a \circ_j b \leqslant_t c$ iff $a \circ_j \sim_t c \leqslant_t \sim_t b$ iff if $j \neq t$, then $a \circ_j \sim_j b \leqslant_t \sim_j c$ (Antilogism);

6) $a \circ_j b \leqslant_t c$ iff $a \leqslant_t b \mapsto_{jt} c$, for each $j, t \leqslant n$.

Let $n > 1$ and $\mathfrak{D}_n = \langle D, \leqslant_1, \ldots, \leqslant_n, \circ_1, \ldots, \circ_n, 1 \rangle$ be a de Morgan multimonoid. The logic $\mathbf{RML}_n$ is built in a propositional language $\mathcal{L}$ with the following connectives: $\neg_j$, $\wedge_j$, $\vee_j$, $\rightarrow_{jt}$, for each $j, t \leqslant n$. These connectives are interpreted in $\mathfrak{D}_n$ as follows (where v is a valuation, i.e. a mapping from the set of all $\mathcal{L}$'s propositional variables to D): $v(A \wedge_j B) = v(A) \cap_j v(B)$, $v(A \vee_j B) = v(A) \cup_j v(B)$, $v(A \rightarrow_{jt} B) = v(A) \mapsto_{jt} v(B)$,[1] $v(\neg_j A) = \sim_j v(A)$. A formula A is $\mathbf{RML}_n$-valid iff $v(A) \geqslant_t 1$, for each de Morgan multimonoid $\mathfrak{D}_n$, each valuation v, and for each $t \leqslant n$.

The axiomatization of the logic $\mathbf{RML}_n$ determined by de Morgan multimonoids is under the development. As a conjecture, we present the following sequent calculus. It uses Slaney's [5] sequent calculus for the positive fragment of $\mathbf{R}$ and the additional rules for negations in the spirit of negation rules used in $\mathbf{ML}_n$ [4, 3] and in Kamide's [2] decidable paraconsistent relevant logic based on the positive fragment of $\mathbf{RW}$. We need some terminological preliminaries.

> "(1) any formula is a bunch, and (2) for $n \geq 2$, if X_i is a bunch for $i = 1, \ldots, n$, then both sequences $(X_1, \ldots, X_n)$ and $(X_1; \ldots; X_n)$ are bunches. Bunches of the forms $(X_1, \ldots, X_n)$ and $(X_1; \ldots; X_n)$ are respectively called *intensional* and *extensional*. Each bunch X_i is called an *immediate constituent* of $(X_1, \ldots, X_n)$ and $(X_1; \ldots; X_n)$. For the sake of simplicity, we assume that immediate constituents of an intensional (and an extensional) bunch are not intensional (and extensional, respectively). Thus, a bunch of the form $(X; (Y; Z); W)$ is identified with the bunch $(X; Y; Z; W)$. In other words, intensional bunches and extensional bunches must appear alternatively in a given bunch." [2, p. 179]

[1] In each de Morgan multimonoid it holds that $a \circ_j b \leqslant_t c$ iff $a \leqslant_t \sim_t(b \circ_j \sim_t c)$. Hence, $a \leqslant_t b \mapsto_{jt} c$ iff $a \leqslant_t \sim_t(b \circ_j \sim_t c)$. Also, we have: $a \leqslant_t b$ iff $1 \leqslant_t a \mapsto_{jt} b$.

In a usual way, one can define the notions of a *subbunch* and an *occurrence of a subbunch* X of Y (such an occurrence is said to be an *indicated bunch occurrence* of X (in Y)). We write $\Gamma(X)$ for an indicated bunch occurrence of X in Γ. The notion of a sequent is understood as an ordered pair written as $X \Rightarrow \varphi$ such that X is a (possibly, empty) bunch and φ is a formula.

Now we are ready to intoduce our sequent calculus. In what follows, j, k, n stands for positive integers such that $j, k \leqslant n$ and $j \neq k$. The axioms are as follows (for any propositional variable p):

$$(\text{Ax})\ p \Rightarrow p \qquad (\text{Ax}_j)\ \neg_j p \Rightarrow \neg_j p$$

The structural rules are as follows:

$$(\text{Cut})\ \frac{X \Rightarrow \varphi \quad \Gamma(\varphi) \Rightarrow \psi}{\Gamma(X) \Rightarrow \psi} \quad (\text{I-ex})\ \frac{\Gamma(X, Y) \Rightarrow \varphi}{\Gamma(Y, X) \Rightarrow \varphi} \quad (\text{I-co})\ \frac{\Gamma(X, X) \Rightarrow \varphi}{\Gamma(X) \Rightarrow \varphi}$$

$$(\text{E-ex})\ \frac{\Gamma(X; Y) \Rightarrow \varphi}{\Gamma(Y; X) \Rightarrow \varphi} \quad (\text{E-co})\ \frac{\Gamma(X; X) \Rightarrow \varphi}{\Gamma(X) \Rightarrow \varphi} \quad (\text{E-wk})\ \frac{\Gamma(X) \Rightarrow \varphi}{\Gamma(X; Y) \Rightarrow \varphi}$$

The non-negated rules are as follows:

$$(\wedge_j \Rightarrow)\ \frac{\Gamma(\varphi; \psi) \Rightarrow \chi}{\Gamma(\varphi \wedge_j \psi) \Rightarrow \chi} \quad (\Rightarrow \wedge_j)\ \frac{X \Rightarrow \varphi \quad Y \Rightarrow \psi}{X; Y \Rightarrow \varphi \wedge_j \psi}$$

$$(\vee_j \Rightarrow)\ \frac{\Gamma(\varphi) \Rightarrow \chi \quad \Gamma(\psi) \Rightarrow \chi}{\Gamma(\varphi \vee_j \psi) \Rightarrow \chi} \quad (\Rightarrow \vee_j)\ \frac{X \Rightarrow \varphi_i}{X \Rightarrow \varphi_1 \vee_j \varphi_2}$$

$$(\rightarrow_{jl}\Rightarrow)\ \frac{X \Rightarrow \varphi \quad \Gamma(\psi) \Rightarrow \chi}{\Gamma(\varphi \rightarrow_{jl} \psi, X) \Rightarrow \chi} \quad (\Rightarrow\rightarrow_{jl})\ \frac{X, \varphi \Rightarrow \psi}{X \Rightarrow \varphi \rightarrow_{jl} \psi}$$

The jj- and jjl-negated logical rules are as follows:

$$(\neg_j\wedge_j \Rightarrow)\ \frac{\Gamma(\neg_j\varphi) \Rightarrow \chi \quad \Gamma(\neg_j\psi) \Rightarrow \chi}{\Gamma(\neg_j(\varphi \wedge_j \psi)) \Rightarrow \chi} \quad (\Rightarrow \neg_j\wedge_j)\ \frac{X \Rightarrow \varphi_i}{X \Rightarrow \neg_j(\varphi_1 \wedge_j \varphi_2)}$$

$$(\neg_j\vee_j \Rightarrow)\ \frac{\Gamma(\neg_j\varphi; \neg_j\psi) \Rightarrow \chi}{\Gamma(\neg_j(\varphi \vee_j \psi)) \Rightarrow \chi} \quad (\Rightarrow \neg_j\vee_j)\ \frac{X \Rightarrow \neg_j\varphi \quad Y \Rightarrow \neg_j\psi}{X; Y \Rightarrow \neg_j(\varphi \vee_j \psi)}$$

$$(\neg_j\neg_j \Rightarrow)\ \frac{\Gamma(\varphi) \Rightarrow \chi}{\Gamma(\neg_j\neg_j\varphi) \Rightarrow \chi} \quad (\Rightarrow \neg_j\neg_j)\ \frac{X \Rightarrow \varphi}{X \Rightarrow \neg_j\neg_j\varphi}$$

$$(\neg_j\rightarrow_{jl} \Rightarrow)\ \frac{\neg_j\psi \Rightarrow Y, \neg_j\varphi}{\neg_j(\varphi \rightarrow_{jl} \psi) \Rightarrow Y} \quad (\Rightarrow \neg_j\rightarrow_{jl})\ \frac{X \Rightarrow \neg_j\psi \quad \Gamma(\neg_j\varphi) \Rightarrow}{\Gamma(X) \Rightarrow \neg_j(\varphi \rightarrow_{jl} \psi)}$$

The ljl-negated logical rules are as follows:

$$(\neg_l\rightarrow_{jl} \Rightarrow)\ \frac{\Gamma(\varphi, \neg_l\psi) \Rightarrow \chi}{\Gamma(\neg_l(\varphi \rightarrow_{jl} \psi)) \Rightarrow \chi} \quad (\Rightarrow \neg_l\rightarrow_{jl})\ \frac{X \Rightarrow \varphi \quad Y \Rightarrow \neg_l\psi}{X, Y \Rightarrow \neg_l(\varphi \rightarrow_{jl} \psi)}$$

The kj-negated logical rules are as follows:

$$(\neg_k\wedge_j \Rightarrow)\ \frac{\Gamma(\neg_k\varphi; \neg_k\psi) \Rightarrow \chi}{\Gamma(\neg_k(\varphi \wedge_j \psi)) \Rightarrow \chi} \quad (\Rightarrow \neg_k\wedge_j)\ \frac{X \Rightarrow \neg_k\varphi \quad Y \Rightarrow \neg_k\psi}{X; Y \Rightarrow \neg_k(\varphi \wedge_j \psi)}$$

$$(\neg_k\vee_j \Rightarrow)\ \frac{\Gamma(\neg_k\varphi) \Rightarrow \chi \quad \Gamma(\neg_k\psi) \Rightarrow \chi}{\Gamma(\neg_k(\varphi \vee_j \psi)) \Rightarrow \chi} \quad (\Rightarrow \neg_k\vee_j)\ \frac{X \Rightarrow \neg_k\varphi_i}{X \Rightarrow \neg_k(\varphi_1 \vee_j \varphi_2)}$$

$$(\neg_k\to_j \Rightarrow)\ \frac{X \Rightarrow \neg_k\varphi \quad \Gamma(\neg_k\psi) \Rightarrow \chi}{\Gamma(\neg_k(\varphi \to_j \psi), X) \Rightarrow \chi} \quad (\Rightarrow \neg_k\to_j)\ \frac{\neg_k\varphi, X \Rightarrow \neg_k\psi}{X \Rightarrow \neg_k(\varphi \to_j \psi)}$$

As for the $\neg_k\neg_j$, we do not offer any rules, but one may find some option. For example, the rules for $\neg_k\neg_j$ can be the same as for $\neg_j\neg_j$ (with the corresponding changes in the algebraic semantics). Atypically for multilattice logic, our calculus has three types (instead of two) of the negation of implication rules. The reason is that not only a multilattice is used, but also a multimonoid.

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