A case study in proof-theoretic tetralateralism

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We present an expansion of the paraconsistent logic **N4** by operators for meaningfulness and nonsensicality. This logic contains three congruentiality-breaking unary connectives, which gives rise to a tetra-lateral sequent calculus with four different sequent arrows.

Keywords: inconsistency-tolerant logic, sequent calculus, proof-theoretic multilateralism.

Joint work with Sara Ayhan

Introduction

In recent years knowledge representation formalisms based on various non-classical logics instead of classical logic have attracted increasing attention. In description logic, for example, various approaches to modeling para-consistent, inconsistency-tolerant reasoning have been developed. In the paper [4, p. 301] from 2003 it was remarked that

> [t]here is some work on description logics using non-monotonic, many-valued, or fuzzy-logic, see [4, Chapter 6], however, the underlying logic of almost all systems of description logic is *classical*.

Here [4] is *The Description Logic Handbook*, edited by F. Baader et al., Cambridge University Press, 2003. When the second part of [4] appeared in 2008 [5], the situation had already changed considerably. In 2005, e.g., the edited volume [2] on inconsistency tolerance was published, observing "a need to develop tolerance to inconsistency in application technologies such as databases, knowledgebases, and software systems," and nowadays inconsistency handling in description logic based ontologies is an established area within knowledge representation.

A prominent example of a paraconsistent logic that has found quite a few applications in knowledge representation and AI is first-degree entailment logic, **FDE**, for a survey see [6]. The system FED lacks a genuine implication, however, and the paraconsistent logic **N4** due to A. Almukdad and D. Nelson [1] expands **FDE** by a constructive conditional. There is a sequent calculus

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for **N4** that makes use of two sequent arrows standing for different derivability relations, one that represents the preservation of support of truth, whereas the other represents the preservation of support of falsity, cf. [3]. The present paper is about an expansion of **N4** by operators for meaningfulness and nonsensicality. This logic contains three congruentiality-breaking unary connectives, which gives rise to a tetralateral sequent calculus with four different sequent arrows.

1. What is proof-theoretic bi- and tetralateralism?

Proof-theoretic tetralateralism is a generalization of proof-theoretic bilateralism. The full paper [10] contains a comparison of various approaches to proof-theoretic bilateralism and proposes a definition of proof-theoretic multilateralism. For reasons of space, in this abstract, we briefly mention only one understanding of proof-theoretic bilateralism and and highlight some differences with the present approach to proof-theoretic tetralateralism.

According to D. Riply, [7], bilateralism "is the view that which inferences are valid is ... to be explained in terms of conditions on *assertion* and *denial*." In [8] he explains that "*bilateralism*, holds that we must consider conditions governing the speech acts of assertion and denial. For a bilateralism to genuinely be *bi*, then, it must hold that denial conditions cannot themselves be understood as deriving only from assertion conditions," and in [9], Ripley characterizes bilateralism as "the view that meanings in general are to be given via conditions on assertion and denial."

Most definitions of bilateralism have in common a reference to the speech acts of assertion and denial, or attitudes of acceptance and rejection, and emphasize that those two notions are on a par and equally important. Another point often mentioned is that in bilateralism rejection or denial are seen as conceptually prior to negation, i.e., the denial of A is not interpreted in terms of, or as the assertion of the negation of A but the other way around

The notion of bilateralism advocated in [10] does not consider speech acts or propositional attitudes as the primary notions to act upon in the context of a proof-theoretic theory of meaning (or semantics in general). Rather proof–refutation, provability–refutability, verification–falsification, demonstrability of meaningfulness–demonstrability of nonsensicality are central pairs of proof-theoretically relevant semantical concepts. This is reflected in the presence of two or more separate derivability relations.

2. A case-study in proof-theoretic tetralateralism

We will expand the language of propositional **N4** by two unary connectives, $[m]$ and $[n]$. A formula $[m]A$ is to be read as "it is meaningful that A", and $[n]A$ is to be understood as "it is nonsensical that A". The logic of the expanded language will be referred to as **N4mn**.

3. Kripke semantics and completeness

The propositional language $\mathcal{L}$ of **N4mn** is defined in Backus-Naur form as follows:

$$\begin{array}{rl} \text{variables } \Phi\text{:} & p \in \Phi \\ \text{formulas:} & A \in \mathit{Form}_{\mathcal{L}}(\Phi) \\ A ::= & p \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid {\sim}A \mid [m]A \mid [n]A. \end{array}$$

We use $A \leftrightarrow B$ as an abbreviation of $(A \to B) \wedge (B \to A)$. The language $\mathcal{L}'$ of positive intuitionistic propositional logic, $\mathbf{IPL}^{+}$, is obtained from $\mathcal{L}$ by dropping the unary connectives, i.e., $\sim$, $[m]$, and $[n]$, and the language $\mathcal{L}''$ of the propositional logic **N4** is obtained from $\mathcal{L}$ by dropping $[m]$ and $[n]$.

Definition 1. *A* Kripke frame *is a structure $\langle M, R\rangle$, where M is a nonempty set (of information states), and R is a reflexive and transitive binary relation (of information state expansion) on M.*

Definition 2. *A* valuation $\models$ *on a Kripke frame $\langle M, R\rangle$ is a mapping from the set Φ of propositional variables to the power set 2^M of M such that for any $p \in \Phi$ and any $x, y \in M$, if $x \in \models(p)$ and xRy, then $y \in \models(p)$. We will write $x \models p$ for $x \in \models(p)$. This valuation $\models$ is extended to a mapping from the set of all $\mathcal{L}'$-formulas to 2^M by:*

$$x \models A \to B \text{ iff } \forall y \in M\ [xRy \text{ and } y \models A \text{ imply } y \models B],$$
$$x \models A \wedge B \text{ iff } x \models A \text{ and } x \models B,$$
$$x \models A \vee B \text{ iff } x \models A \text{ or } x \models B.$$

If $\mathcal{F} = \langle M, R\rangle$ is a Kripke frame, then $\langle M, R, \models\rangle$ is a Kripke model for $\mathbf{IPL}^{+}$ based on $\mathcal{F}$.

The following *heredity condition* holds for $\models$: for any $\mathcal{L}'$-formula A and any $x, y \in M$, if $x \models A$ and xRy, then $y \models A$.

Definition 3. *An $\mathcal{L}'$-formula A is* true in a Kripke model $\langle M, R, \models\rangle$ *for* $\mathbf{IPL}^{+}$ *if $x \models A$ for any $x \in M$, and is* valid on a Kripke frame $\mathcal{F} = \langle M, R\rangle$ *if it is true for every Kripke model for $\mathbf{IPL}^{+}$ based on $\mathcal{F}$. An $\mathcal{L}'$-formula A is said to be* $\mathbf{IPL}^{+}$-valid *if A is valid on every Kripke frame. Let $\Gamma \cup \{A\}$ be a* *set of $\mathcal{L}'$-formulas. Semantic consequence (entailment) is defined in terms of truth preservation at each state:* $\Gamma \models A$ *if for every Kripke model* $\langle M, R, \models \rangle$ *for* $\mathbf{IPL}^+$ *and for all* $x \in M$, $x \models A$ *if* $x \models B$ *for all* $B \in \Gamma$. *We define the logic* $\mathbf{IPL}^+$ *model-theoretically as the pair* $\langle \mathcal{L}', \{\Gamma, A \mid \Gamma \models A\}\rangle$.

We turn to the language $\mathcal{L}$ and define *four* separate valuation functions $\models^+$, $\models^-$, $\models^m$, and $\models^n$. These mappings determine for a given propositional variable p, the set of states that support the truth, the falsity, the meaningfulness, and the nonsensicality (meaninglessness) of p, respectively. Support of truth, support of falsity, support of meaningfulness, and support of meaninglessness are seen as properties that are independent of each other. In particular, it is not excluded that an information state supports both the truth and the falsity of a given propositional variable or both its meaningfulness and its nonsensicality.

Definition 4. *The valuation functions* $\models^+$, $\models^-$, $\models^m$, *and* $\models^n$ *on a Kripke frame* $\langle M, R\rangle$ *are mappings from the set* Φ *to the power set* 2^M *of* M *such that for any* $\star \in \{+, -, m, n\}$, *any* $p \in \Phi$ *and any* $x, y \in M$, *if* $x \in \models^\star (p)$ *and* xRy, *then* $y \in \models^\star (p)$. *We will write* $x \models^\star p$ *for* $x \in \models^\star (p)$. *The functions* $\models^+$, $\models^-$, $\models^m$, *and* $\models^n$ *are extended to mappings from the set of all formulas to* 2^M *by:*

1)
$x \models^+ A \wedge B$ *iff* $x \models^+ A$ *and* $x \models^+ B$,
$x \models^+ A \vee B$ *iff* $x \models^+ A$ *or* $x \models^+ B$,
$x \models^+ A \to B$ *iff* $\forall y \in M\ [xRy$ *and* $y \models^+ A$ *imply* $y \models^+ B]$,
$x \models^+ {\sim}A$ *iff* $x \models^- A$,
$x \models^+ [m]A$ *iff* $x \models^m A$,
$x \models^+ [n]A$ *iff* $x \models^n A$,

2)
$x \models^- A \wedge B$ *iff* $x \models^- A$ *or* $x \models^- B$,
$x \models^- A \vee B$ *iff* $x \models^- A$ *and* $x \models^- B$,
$x \models^- A \to B$ *iff* $x \models^+ A$ *and* $x \models^- B$,
$x \models^- {\sim}A$ *iff* $x \models^+ A$,
$x \models^- [m]A$ *iff* $x \models^n A$,
$x \models^- [n]A$ *iff* $x \models^m A$,

3)
$x \models^m A \circ B$ *iff* $x \models^m A$ *and* $x \models^m B$, *for* $\circ \in \{\wedge, \vee, \to\}$,
$x \models^m \circ A$ *iff* $x \models^m A$, *for* $\circ \in \{\sim, [m], [n]\}$,

4)
$x \models^n A \circ B$ *iff* $x \models^n A$ *or* $x \models^n B$, *for* $\circ \in \{\wedge, \vee, \to\}$,
$x \models^n \circ A$ *iff* $x \models^n A$, *for* $\circ \in \{\sim, [m], [n]\}$.

If $\mathcal{F} = \langle M, R\rangle$ *is a Kripke frame, then* $\langle M, R, \models^+, \models^-, \models^m, \models^n\rangle$ *is a* Kripke model *for* **N4mn** *based on* $\mathcal{F}$.

The heredity condition holds for $\models^+$, $\models^-$, $\models^m$, and $\models^n$, i.e., for any $\mathcal{L}$-formula A and any $x, y \in M$, if $x \models^* A$ and xRy, then $y \models^* A$, for $* \in \{+, -, m, n\}$.

As to a motivation of the semantical clauses for $[m]$ and $[n]$, we may note that a compound formula is meaningful (nonsensical) iff all (some) of its immediate proper subformulas are; meaninglessness is 'infectious'. Thus, in particular, $x \models^m [n]A$ iff $x \models^m A$, and $x \models^m [n]A$ does not, in general, imply $x \models^+ [n]A$. For the statement that A is nonsensical to be meaningful, A must be meaningful, although $[n]A$ may well be false.

Definition 5. *An $\mathcal{L}$-formula A is said to be* true *in a Kripke model for* **N4mn** $\langle M, R, \models^+, \models^-, \models^m, \models^n \rangle$ *if $x \models^+ A$ for any $x \in M$, and to be valid on a Kripke frame $\mathcal{F} = \langle M, R \rangle$ if it is true for every Kripke model for* **N4mn** *based on $\mathcal{F}$. An $\mathcal{L}$-formula A is said to be* **N4mn***-valid if A is valid on every Kripke frame. Let $\Gamma \cup \{A\}$ be a set of $\mathcal{L}$-formulas. Entailment is defined in terms of support-of-truth preservation at each state: $\Gamma \models^+ A$ if for all Kripke models for* **N4mn** $\langle M, R, \models^+, \models^-, \models^m, \models^n \rangle$ *and for all $x \in M$, $x \models^+ A$ if $x \models^+ B$ for all $B \in \Gamma$. We write $A \models^+ B$ for $\{A\} \models^+ B$. We define the logic* **N4mn** *model-theoretically as the pair $\langle \mathcal{L}, \{\Gamma, A \mid \Gamma \models^+ A\} \rangle$ and* **N4** *is model-theoretically defined as $\langle \mathcal{L}'', \{\Gamma, A \mid \Gamma \models^+ A\} \rangle$.*

Proposition 1. *Each of the unary connectives $\circ \in \{\sim, [m], [n]\}$ is congruentiality-breaking in the sense that there are $\mathcal{L}$-formulas A and B such that $A \models^+ B$ and $B \models^+ A$ but not: $\circ A \models^+ \circ B$ and $\circ B \models^+ \circ A$.*

Definition 6. *Given the set Φ of propositional variables, we define three more sets of propositional variables, namely $\Phi^- := \{p^- \mid p \in \Phi\}$, $\Phi^m := \{p^m \mid p \in \Phi\}$, and $\Phi^n := \{p^n \mid p \in \Phi\}$. We inductively define a mapping f from $\mathrm{Form}_{\mathcal{L}}(\Phi)$ to the set of formulas of the language $\mathcal{L}'$ of* **IPL**$^+$ *defined over $\Phi \cup \Phi^- \cup \Phi^m \cup \Phi^n$ as follows:*

1) *for any $p \in \Phi$, $f(p) := p$, $f(\sim p) := p^-$, $f([m]p) := p^m$, $f([n]p) := p^n$,*
2) *$f(A \circ B) := f(A) \circ f(B)$ for $\circ \in \{\rightarrow, \wedge, \vee\}$,*
3) *$f(\sim(A \wedge B)) := f(\sim A) \vee f(\sim B)$,*
4) *$f(\sim(A \vee B)) := f(\sim A) \wedge f(\sim B)$,*
5) *$f(\sim(A \rightarrow B)) := f(A) \wedge f(\sim B)$,*
6) *$f(\sim\sim A) := f(A)$,*
7) *$f(\sim[m]A) := f([n]A)$,*
8) *$f(\sim[n]A) := f([m]A)$,*
9) *$f([m](A \circ B)) := f([m]A) \wedge f([m]B)$, for $\circ \in \{\rightarrow, \wedge, \vee\}$,*
10) *$f([m] \circ A) := f([m]A)$, for $\circ \in \{\sim, [m], [n]\}$,*
11) *$f([n](A \circ B)) := f([n]A) \vee f([n]B)$, for $\circ \in \{\rightarrow, \wedge, \vee\}$,*

12) $f([n] \circ A) := f([n]A)$, *for* $\circ \in \{\sim, [m], [n]\}$.

We write $f(\Gamma)$ *to denote the result of replacing every occurrence of a formula* A *in* Γ *by an occurrence of* $f(A)$*; thus,* $f(\varnothing) = \varnothing$.

Lemma 1. *Let* f *be the function defined in Definition 6. For any Kripke model for* **N4mn** $\langle M, R, \models^+, \models^-, \models^m, \models^n\rangle$*, we can define a Kripke model for* $\mathbf{Int}^+$ $\langle M, R, \models\rangle$ *such that for any* $A \in \mathrm{Form}_{\mathcal{L}}(\Phi)$ *and any* $x \in M$*,*

(1) $x \models^+ A$ *iff* $x \models f(A)$,
(2) $x \models^- A$ *iff* $x \models f(\sim A)$,
(3) $x \models^m A$ *iff* $x \models f([m]A)$,
(4) $x \models^n A$ *iff* $x \models f([n]A)$.

Lemma 2. *Let* f *be the function defined in Definition 6. For any Kripke model* $\langle M, R, \models\rangle$ *for* $\mathbf{IPL}^+$*, we can construct a Kripke model* $\langle M, R, \models^+, \models^-, \models^m, \models^n\rangle$ *for* **N4mn** *such that for any* $\mathcal{L}$*-formula* A *and any* $x \in M$*,*

1) $x \models f(A)$ *iff* $x \models^+ A$,
2) $x \models f(\sim A)$ *iff* $x \models^- A$,
3) $x \models f([m]A)$ *iff* $x \models^m A$,
4) $x \models f([n]A)$ *iff* $x \models^n A$.

Theorem 1 (Semantical embedding)**.** *Let* f *be the mapping from Definition 6. For any set of* $\mathcal{L}$*-formulas* $\Gamma \cup A$, $\Gamma \models^+ A$ *in* **N4mn** *iff* $f(\Gamma) \models f(A)$ *in* $\mathbf{IPL}^+$.

3.1. A tetralateral sequent calculus for N4mn

We define a tetralateral sequent calculus SN4mn for **N4mn** that makes use of four different sequent arrows by generalizing a combination of the sequent calculi Sn4 and Dn4 from [3]. A sequent is an expression of the form,

$$\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* A$$

where $\Gamma_1, \ldots, \Gamma_4$ are finite, possibly empty multisets of $\mathcal{L}$-formulas, A is an $\mathcal{L}$-formula, and $* \in \{+, -, m, n\}$. For a singleton multiset $\{A\}$ we usually write just A, and A, Γ as well as Γ, A (Δ, Γ as well as Γ, Δ) designates the union of the multisets Γ and $\{A\}$ (Δ and Γ).

Definition 7. *Let* $* \in \{+, -, m, n\}$, $\circ \in \{\sim, [m], [n]\}$*, and* $\sharp \in \{\wedge, \vee, \rightarrow\}$*. The sequent calculus* SN4mn *is given by the following sequents and sequent rules. The axiomatic sequents of* SN4mn *are of the form:*

$$p : \varnothing : \varnothing : \varnothing \Rightarrow^- p \qquad \varnothing : p : \varnothing : \varnothing \Rightarrow^+ p$$
$$\varnothing : \varnothing : p : \varnothing \Rightarrow^m p \qquad \varnothing : \varnothing : \varnothing : p \Rightarrow^n p$$

for any $p \in \Phi$, where $\varnothing$ is the empty multiset.
The structural rules of SN4mn *are of the form:*

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- A \qquad A, \Gamma'_1 : \Gamma'_2 : \Gamma'_3 : \Gamma'_4 \Rightarrow^* C}{\Gamma_1, \Gamma'_1 : \Gamma_2, \Gamma'_2 : \Gamma_3, \Gamma'_3 : \Gamma_4, \Gamma'_4 \Rightarrow^* C}\ (\mathrm{cut}-)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ A \qquad \Gamma'_1 : A, \Gamma'_2 : \Gamma'_3 : \Gamma'_4 \Rightarrow^* C}{\Gamma_1, \Gamma'_1 : \Gamma_2, \Gamma'_2 : \Gamma_3, \Gamma'_3 : \Gamma_4, \Gamma'_4 \Rightarrow^* C}\ (\mathrm{cut}+)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^m A \qquad \Gamma'_1 : \Gamma'_2 : A, \Gamma'_3 : \Gamma'_4 \Rightarrow^* C}{\Gamma_1, \Gamma'_1 : \Gamma_2, \Gamma'_2 : \Gamma_3, \Gamma'_3 : \Gamma_4, \Gamma'_4 \Rightarrow^* C}\ (\mathrm{cut}m)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^n A \qquad \Gamma'_1 : \Gamma'_2 : \Gamma'_3 : A, \Gamma'_4 \Rightarrow^* C}{\Gamma_1, \Gamma'_1 : \Gamma_2, \Gamma'_2 : \Gamma_3, \Gamma'_3 : \Gamma_4, \Gamma'_4 \Rightarrow^* C}\ (\mathrm{cut}n)$$

$$\frac{A, A, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{A, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\mathrm{co}-) \qquad \frac{\Gamma_1 : A, A, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{\Gamma_1 : A, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\mathrm{co}+)$$

$$\frac{\Gamma_1 : \Gamma_2 : A, A, \Gamma_3 : \Gamma_4 \Rightarrow^* C}{\Gamma_1 : \Gamma_2 : A, \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\mathrm{co}m) \qquad \frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : A, A, \Gamma_4 \Rightarrow^* C}{\Gamma_1 : \Gamma_2 : \Gamma_3 : A; \Gamma_4 \Rightarrow^* C}\ (\mathrm{co}n)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{A, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\mathrm{we}-) \qquad \frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{\Gamma_1 : A, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\mathrm{we}+)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{\Gamma_1 : \Gamma_2 : A, \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\mathrm{we}m) \qquad \frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{\Gamma_1 : \Gamma_2 : \Gamma_3 : A, \Gamma_4 \Rightarrow^* C}\ (\mathrm{we}n).$$

The introduction rules for unary connectives in succedent position of sequents are of the form:

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- A}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ {\sim}A}\ ({\sim}\mathrm{r}+) \qquad \frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ A}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- {\sim}A}\ ({\sim}\mathrm{r}-)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^m A}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ [m]A}\ ([m]\mathrm{r}+) \qquad \frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^n A}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- [m]A}\ ([m]\mathrm{r}-)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^n A}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ [n]A}\ ([n]\mathrm{r}+) \qquad \frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^m A}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- [n]A}\ ([n]\mathrm{r}-)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^m A}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^m \circ A}\ (\circ\mathrm{r}m) \qquad \frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^n A}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^n \circ A}\ (\circ\mathrm{r}n).$$

The introduction rules for unary connectives in antecedent position of sequents are of the form:

$$\frac{A, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{\Gamma_1 : {\sim}A, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ ({\sim}\mathrm{l}+) \qquad \frac{\Gamma_1 : A, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{{\sim}A, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ ({\sim}\mathrm{l}-)$$

$$\frac{\Gamma_1 : \Gamma_2 : A, \Gamma_3 : \Gamma_4 \Rightarrow^* C}{\Gamma_1 : [m]A, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ ([m]\mathrm{l}+) \qquad \frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : A, \Gamma_4 \Rightarrow^* C}{[m]A, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ ([m]\mathrm{l}-)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : A, \Gamma_4 \Rightarrow^* C}{\Gamma_1 : [n]A, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ ([n]\mathrm{l}+) \quad \frac{\Gamma_1 : \Gamma_2 : A, \Gamma_3 : \Gamma_4 \Rightarrow^* C}{[n]A, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ ([n]\mathrm{l}-)$$

$$\frac{\Gamma_1 : \Gamma_2 : A, \Gamma_3 : \Gamma_4 \Rightarrow^* C}{\Gamma_1 : \Gamma_2 : \circ A, \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\circ\mathrm{l}m) \quad \frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : A, \Gamma_4 \Rightarrow^* C}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \circ A, \Gamma_4 \Rightarrow^* C}\ (\circ\mathrm{l}n).$$

The positive inference rules for the binary connectives of SN4mn *are of the form:*

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ A \qquad \Gamma'_1 : B, \Gamma'_2 : \Gamma'_3 : \Gamma'_4 \Rightarrow^* C}{\Gamma_1, \Gamma'_1 : A \to B, \Gamma_2, \Gamma'_2 : \Gamma_3, \Gamma'_3 : \Gamma_4, \Gamma'_4 \Rightarrow^* C}\ (\to \mathrm{l}+)$$

$$\frac{\Gamma_1 : A, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ B}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ A \to B}\ (\to \mathrm{r}+)$$

$$\frac{\Gamma_1 : A, B, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{\Gamma_1 : A \wedge B, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\wedge\mathrm{l}+)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ A \quad \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ B}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ A \wedge B}\ (\wedge\mathrm{r}+)$$

$$\frac{\Gamma_1 : A \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C \qquad \Gamma_1 : B, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{\Gamma_1 : A \vee B, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\vee\mathrm{l}+)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ A}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ A \vee B}\ (\vee\mathrm{r}1+) \quad \frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ B}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ A \vee B}\ (\vee\mathrm{r}2+).$$

The negative inference rules for the binary connectives of SN4mn *are of the form:*

$$\frac{B, \Gamma_1 : A, \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{A \to B, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\to \mathrm{l}-)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ A \qquad \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- B}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- A \to B}\ (\to \mathrm{r}-)$$

$$\frac{A, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C \qquad B, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{A \wedge B, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\wedge\mathrm{l}-)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- A}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- A \wedge B}\ (\wedge\mathrm{r}1-) \quad \frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- B}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- A \wedge B}\ (\wedge\mathrm{r}2-)$$

$$\frac{A, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{A \vee B, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\vee\mathrm{l}1-) \quad \frac{B, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}{A \vee B, \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\vee\mathrm{l}2-)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- A \qquad \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- B}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- A \vee B}\ (\vee\mathrm{r}-).$$

The m-related inference rules for the binary connectives of SN4mn *are of the form:*

$$\frac{\Gamma_1 : \Gamma_2 : A, B, \Gamma_3 : \Gamma_4 \Rightarrow^* C}{\Gamma_1 : \Gamma_2 : A\sharp B, \Gamma_3 : \Gamma_4 \Rightarrow^* C}\ (\sharp\mathrm{l}m)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^m A \qquad \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^m B}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^m A\sharp B}\ (\sharp\mathrm{r}m).$$

The n*-related inference rules for the binary connectives of* SN4mn *are of the form:*

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : A, \Gamma_4 \Rightarrow^* C \qquad \Gamma_1 : \Gamma_2 : \Gamma_3 : B, \Gamma_4 \Rightarrow^* C}{\Gamma_1 : \Gamma_2 : \Gamma_3 : A\sharp B, \Gamma_4 \Rightarrow^* C}\ (\sharp\mathrm{l}n)$$

$$\frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^n A}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^n A\sharp B}\ (\sharp\mathrm{r}1n) \qquad \frac{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^n B}{\Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^n A\sharp B}\ (\sharp\mathrm{r}2n).$$

Proposition 2. *In* SN4mn*, for any* $\mathcal{L}$*-formula* A*,*

1. $\vdash A : \varnothing : \varnothing : \varnothing \Rightarrow^- A$,
2. $\vdash \varnothing : A : \varnothing : \varnothing \Rightarrow^+ A$,
3. $\vdash \varnothing : \varnothing : A : \varnothing \Rightarrow^m A$,
4. $\vdash \varnothing : \varnothing : \varnothing : A \Rightarrow^n A$.

3.2. Syntactical embedding, cut-elimination, decidability, and completeness

We syntactically embed SN4mn into Gentzen's sequent calculus LJ^+ for $\mathbf{IPL}^+$. From this embedding we obtain the admissibility of SN4mn's cut-rules, the decidability of SN4mn, and its completeness with respect to the class of all models for **N4mn**. A sequent of LJ^+ is an ordinary sequent, i.e., an expression of the form $\Gamma \Rightarrow A$ where Γ is a finite multiset of $\mathcal{L}'$-formulas and A is an $\mathcal{L}'$-formula. We consider $\mathcal{L}'$ defined over $\Phi \cup \Phi^- \cup \Phi^m \cup \Phi^n$.

Theorem 2 (Syntactical embedding)**.** *Let* f *be the mapping from Definition 6. For any finite multiset of* $\mathcal{L}$*-formulas* $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \{A\}$ *we have:*

(a) (1) $\vdash \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ A$ *in* SN4mn *iff*
$\vdash f({\sim}\Gamma_1), f(\Gamma_2), f([m]\Gamma_3), f([n]\Gamma_4) \Rightarrow f(A)$ *in* LJ^+*;*

(2) $\vdash \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- A$ *in* SN4mn *iff*
$\vdash f({\sim}\Gamma_1), f(\Gamma_2), f([m]\Gamma_3), f([n]\Gamma_4) \Rightarrow f({\sim}A)$ *in* LJ^+*;*

(3) $\vdash \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^m A$ *in* SN4mn *iff*
$\vdash f({\sim}\Gamma_1), f(\Gamma_2), f([m]\Gamma_3), f([n]\Gamma_4) \Rightarrow f([m]A)$ *in* LJ^+*;*

(4) $\vdash \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^n A$ *in* SN4mn *iff*
$\vdash f({\sim}\Gamma_1), f(\Gamma_2), f([m]\Gamma_3), f([n]\Gamma_4) \Rightarrow f([n]A)$ *in* LJ^+.

(b) (1) $\vdash \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^+ A$ *in* SN4mn$-\{(\mathrm{cut}+), (\mathrm{cut}-), (\mathrm{cut}m), (\mathrm{cut}n)\}$ *iff*
$\vdash f({\sim}\Gamma_1), f(\Gamma_2), f([m]\Gamma_3), f([n]\Gamma_4) \Rightarrow f(A)$ *in* $\mathrm{LJ}^+ - (\mathrm{cut})$*;*

(2) $\vdash \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^- A$ *in* SN4mn$-\{(\mathrm{cut}+), (\mathrm{cut}-), (\mathrm{cut}m), (\mathrm{cut}n)\}$ *iff*
$\vdash f({\sim}\Gamma_1), f(\Gamma_2), f([m]\Gamma_3), f([n]\Gamma_4) \Rightarrow f({\sim}A)$ *in* $\mathrm{LJ}^+ - (\mathrm{cut})$*;*

(3) $\vdash \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^m A$ *in* SN4mn$-\{(\mathrm{cut}+), (\mathrm{cut}-), (\mathrm{cut}m), (\mathrm{cut}n)\}$ *iff*
$\vdash f({\sim}\Gamma_1), f(\Gamma_2), f([m]\Gamma_3), f([n]\Gamma_4) \Rightarrow f([m]A)$ *in* $\mathrm{LJ}^+ - (\mathrm{cut})$*;*

(4) $\vdash \Gamma_1 : \Gamma_2 : \Gamma_3 : \Gamma_4 \Rightarrow^n A$ *in* SN4mn$-\{(\text{cut}+), (\text{cut}-), (\text{cut}m), (\text{cut}n)\}$ *iff*
$\vdash f({\sim}\Gamma_1), f(\Gamma_2), f([m]\Gamma_3), f([n]\Gamma_4) \Rightarrow f([n]A)$ *in* LJ$^+$ $-$ (cut).

Theorem 3 (Cut-admissibility)**.** *The rules* (cut+)*,* (cut$-$)*,* (cut$m$)*, and* (cut$n$) *are admissible in cut-free* SN4mn*.*

As a corollary to cut-admissibility one obtains the subformula property for SN4mn, i.e., if a sequent s is provable in SN4mn, then there is a proof π of s such that all formulas appearing in π are subformulas of some formula in s. Moreover, by the decidability of LJ$^+$, for each $\mathcal{L}$-formula A, it is possible to decide whether $f(A)$ is provable in LJ$^+$. Then, by the syntactical embedding theorems, SN4mn is decidable.

Theorem 4 (Completeness)**.** *Let* $\Gamma = \{A_1, \ldots, A_n\} \cup \{A\}$ *be a set of* $\mathcal{L}$*-formulas, let* $\bigwedge\Gamma$ *be the conjunction* $(\ldots(A_1 \wedge A_2) \wedge \ldots \wedge A_n)$*, and let* $\bigwedge\varnothing$ *be the formula* $p \rightarrow p$ *for some fixed* $p \in \Phi$*. Then* $\Gamma \models^+ A$ *in* **N4mn** *iff* $\varnothing : \varnothing : \varnothing : \varnothing \Rightarrow^+ \bigwedge\Gamma \rightarrow A$ *is provable in* SN4mn*.*

References

[1] A. Almukdad and D. Nelson, "Constructible Falsity and Inexact Predicates", *J. Symbolic Logic*, **49** (1984), 231–233.

[2] *Inconsistency Tolerance*, LNCS 3300, eds. L. Bertossi, A.Hunter, Torsten Schaub, Springer, 2005.

[3] N. Kamide and H. Wansing, "Proof Theory of Nelson's Paraconsistent Logic: A Uniform Perspective", *Theoretical Computer Sci.*, **6** (2012), 1–38.

[4] S.P. Odintsov and H. Wansing, "Inconsistency-tolerant Description Logic. Motivation and Basic Systems", *Trends in Logic. 50 Years of Studia Logica*, eds. V. Hendricks, J. Malinowski, Kluwer, Dordrecht, 2003, 301–335.

[5] S.P. Odintsov and H. Wansing, "Inconsistency-tolerant Description Logic. Part II: Tableau Algorithms", *J. Applied Logic*, **6** (2008), 343–360.

[6] H. Omori and H. Wansing, "40 years of FDE: An Introductory Overview", *Studia Logica*, **105** (2017), 1021–1049.

[7] D. Ripley, "Paradoxes and Failures of Cut", *Australasian J. Philosophy*, **91** (2013), 139-164.

[8] D. Ripley, "Bilateralism, Coherence, Warrant", *Act-Based Conceptions of Propositional Content: Contemporary and Historical Perspectives*, eds. F. Moltmann, M. Textor, OUP, 2017, 307–324.

[9] D. Ripley, "Denial", *The Oxford Handbook of Negation*, eds. V. Déprez, M.T. Espinal, OUP, 2020, 47–57.

[10] H. Wansing and S. Ayhan, "Logical multilateralism", *submitted*, 2021.