Remarks on the Definition of Cellular Automaton with Locators [1]

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In [1], a cellular automaton with locators is defined. In this paper we indicate some inaccuracies and issues of this definition and clarify it to get rid of these issues. We also give examples of cellular automata classes with locators that have good properties in a certain sense.

Keywords: cellular automata, homogeneous structures.

1. Introduction

The concept of a cellular automaton (CA) with locators was introduced in [1]. CA with locators is defined by a 8-tuple $(\mathbb{Z}^n, Q, V, E, +, L, \varphi, \psi)$. CA with locators in comparison with a conventional CA $(\mathbb{Z}^n, Q, V, \varphi)$ contains additional structure where different elementary automata (*cells*) can broadcast signals from the set of *broadcasting signals* E computed by the *broadcasting function* ψ. The signals are summed with the commutative semigroup operation $+$. Locators of each cell receive the sum of signals from the directions specified by the solid angles from the set L. CA with locators can be considered as a mathematical model of a device where there are both local interactions between adjacent cells and non-local interactions through broadcasting, which can be implemented using some kind of substrate that sums the signals from the cells due to some physical principle. Such devices can potentially solve some problems in a more natural way than conventional cellular automata, where sometimes we need to develop complicated algorithms, in particular when we need to transmit control signals.

2. Definition of a cellular automaton with locators according to Gasanov

Let us recall the definition of a cellular automaton with locators introduced by E. E. Gasanov in [1].

By a *solid angle* in $\mathbb{R}^k$ we mean the union of all the rays in the space $\mathbb{R}^k$ emanating from a given point (*vertex of an angle*) and intersecting some

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hypersurface in $\mathbb{R}^k$. In the definition, we assume that a solid angle does not contain its vertex. In particular, in this paper we consider two degenerate cases: the full solid angle coinciding with $\mathbb{R}^k$ without the vertex of the angle, which we denote by Ω, and solid angles equal to one ray. If a solid angle is a ray, we denote it by a vector defining its direction.

A *cellular automaton with locators* is a 8-tuple

$$\sigma = (\mathbb{Z}^k, E_n, V, E_q, +, L, \varphi, \psi)$$

where $\mathbb{Z}^k$ is the set of k-dimensional vectors with integer coordinates, $E_n = \{0, 1, \dots, n-1\}$, $V = (\alpha_1, \dots, \alpha_{h-1})$ is an ordered set of pairwise different nonzero vectors from $\mathbb{Z}^k$, $E_q = \{0, 1, \dots, q-1\}$, $+$ is a commutative semigroup operation defined on E_q, $L = (\nu_1, \dots, \nu_m)$ is an ordered set of pairwise different solid angles in $\mathbb{R}^k$ with a vertex at the origin, $\varphi : E_n^h \times E_q^m \to E_n$ is a function depending on the variables $x_0, x_1, \dots, x_{h-1}, z_1, \dots, z_m$ such that $\varphi(0, \dots, 0) = 0$, $\psi : E_n^h \times E_q^m \to E_q$ is a function that depends on the variables $x_0, x_1, \dots, x_{h-1}, z_1, \dots, z_m$. Here the variables $x_0, x_1, \dots, x_{h-1}$ take values from E_n and the variables $z_1, \dots, z_m$ take values from E_q. Elements of the set $\mathbb{Z}^k$ are called *cells* of the cellular automaton σ; elements of the set E_n are called *cell states* of the cellular automaton σ; the set V is called the *neighborhood pattern* of the cellular automaton σ; elements of the set E_q are called *broadcasting signals*; the set L is called the *locator pattern* of the cellular automaton σ; the function φ is called the *local transition function* of the automaton σ; the function ψ is called the *broadcasting function* of the automaton σ. The state 0 is interpreted as *rest state* and the condition $\varphi(0, \dots, 0) = 0$ is interpreted as a condition for maintaining the rest state.

Here we need to introduce an ordering of the neighborhood pattern V and the locator pattern L in order to establish a one-to-one correspondence between vectors from V and solid angles from L and variables x_0, $x_1, \dots, x_{h-1}$, $z_1, \dots, z_m$ of the local transition function φ and the broadcasting function ψ respectively. We can make this correspondence more explicit if we index the variables of the functions φ and ψ by the vectors and solid angles themselves, i.e. assume that the local transition function φ and the broadcasting function ψ depend on the variables $x_{\mathbf{0}}$, $x_{\alpha_1}, \dots, x_{\alpha_{h-1}}$, $z_{\nu_1}, \dots, z_{\nu_m}$, where the index of the first variable is the zero vector $\mathbf{0} = (0, \dots, 0) \in \mathbb{Z}^k$. If we index the variables of the local transition function and broadcasting function in this way, we can write them in any order, and then we can define the neighborhood pattern and the locator pattern simply as a set, not an ordered set.

In the rest of this section we use these conventions: consider the neighborhood pattern as a set of vectors, and the locator pattern as a set of solid angles and index the variables of the local transition function and broadcasting function by the vectors from the neighborhood pattern and solid angles from the locator pattern. At the same time, we often omit the outer parentheses of the vectors in the indices. For example, if $k = 2$, $n = 2$, $q = 2$, $V = \{(-1,0),(1,0)\}$, and $L = \{\Omega,(0,1)\}$, then a local transition function may look like this: $\varphi = x_{-1,0}\&z_\Omega \vee x_{1,0}\&z_{0,1}$.

If $\alpha \in \mathbb{Z}^k$, ν is a solid angle with vertex at the origin, then by $\nu(\alpha)$ we denote the solid angle obtained by translation of the angle ν to the point α.

If $\alpha \in \mathbb{Z}^k$ is a cell of a cellular automaton with locators σ, then the set $V(\alpha) = \{\alpha, \alpha+\alpha_1, \ldots, \alpha+\alpha_{h-1}\}$ is called the *neighborhood of the cell* α, and elements of the set $L(\alpha) = \{\nu_1(\alpha), \ldots, \nu_m(\alpha_m)\}$ are called *locators of the cell* α.

A *state of a cellular automaton with locators* σ is a pair (e, f), where e is an arbitrary function from the set $\mathbb{Z}^k$ to the set E_q, called *broadcast state*, f is an arbitrary function from the set $\mathbb{Z}^k$ to the set E_n and called *distribution of states of the cellular automaton with locators* σ. Such a function can be interpreted as a certain mosaic arising in the k-dimensional space as a result of assigning a certain state from the set E_n and some signal from the set E_q to each point with integer coordinates. The set of all possible states of a cellular automaton with locators is denoted by Σ.

If $\alpha \in \mathbb{Z}^k$ and (e, f) is a state of a cellular automaton with locators σ, then the value $e(\alpha)$ is called *the signal of the cell* α*, defined by the state* (e, f), and the value $f(\alpha)$ is *the state of the cell* α*, determined by the state* (e, f). For each $i \in \{1, \ldots, m\}$ the value

$$s_i(\alpha) = \sum_{\beta \in \nu_i(\alpha) \cap \mathbb{Z}^k} e(\beta) \tag{1}$$

we call *the value of the locator* ν_i*, determined by the state* (e, f). Here, in the summation the semigroup operation $+$ defined on E_q is used.

On the set Σ we define the *global transition function* Φ of a cellular automaton with locators σ, putting $\Phi(e, f) = (e', f')$, where $(e, f), (e', f') \in \Sigma$ and for any cell $\alpha \in \mathbb{Z}^k$ the following identities hold

$$f'(\alpha) = \varphi(f(\alpha), f(\alpha+\alpha_1), \ldots, f(\alpha+\alpha_{h-1}), s_1(\alpha), \ldots, s_m(\alpha)), \tag{2}$$

$$e'(\alpha) = \psi(f(\alpha), f(\alpha+\alpha_1), \ldots, f(\alpha+\alpha_{h-1}), s_1(\alpha), \ldots, s_m(\alpha)). \tag{3}$$

A meaningful interpretation of the mapping Φ is that the signal of each cell and the state of each cell "after the transition" is determined by the state of the neighborhood of the cell and by the values of the locators "before the transition" using the rules ψ and φ in the same way for all cells.

By the *behavior of a cellular automaton with locators* σ we call a sequence $(e_0, f_0), (e_1, f_1), (e_2, f_2), \ldots$ of states such that the equation $(e_{i+1}, f_{i+1}) = \Phi(e_i, f_i)$ holds for all $i = 0, 1, 2, \ldots$. The state (e_i, f_i) is called the *state of* *the cellular automaton with locators σ at the time i*, and (e_0, f_0) is also called the *initial state of the cellular automaton with locators σ*.

A state of a cellular automaton is called a *configuration* if only a finite number of cells is in a state other than 0 and the signals of all cells are zero. The set of configurations is denoted by Σ'.

If a certain state of a cellular automaton is specified, then cells that are in a state other than 0 are called *active*.

3. Corrections for the definition

3.1. Restriction on solid angles

According to the definition in Section 2, a solid angle is a union of rays intersecting some hypersurface. However, even in the two-dimensional case, an angle is defined by a real number which can be used to encode an infinite amount of information. For the two-dimensional case, we propose to restrict the set of solid angles to the set of angles bounded by rays going through points with rational coefficients.

For the multidimensional case, there is even more freedom of choice of a solid angle. In this case, we propose to introduce the following restriction: the boundary of a solid angle should consist of hyperplanes spanned by points with integer coordinates. Note that degenerate solid angles completely contained in a subspace of a lower dimension are also allowed. Boundary of such a degenerate angle should consist of parts of hyperplanes specified by linear equations with integer coefficients.

3.2. Restrictions on the semigroup and the broadcasting function

The definition of cellular automaton requires the existence of a distinguished zero state which is preserved by the transition function. It is natural to add a similar requirement for the broadcasting alphabet. Formally, in [1] the set E always has a form $\{0, ..., q-1\}$ and contains 0, however, there is no requirement that $0 + x = 0$. We propose not to require that E has the form $\{0, ..., q-1\}$ but could contain elements of arbitrary type (apart from numbers, it is often convenient to use pairs or sets of numbers), but require that the semigroup $(E, +)$ is a monoid, i.e. there exists a neutral element $0 \in E$ such that $0 + x = x$ for all $x \in E$.

In [1] there is a restriction on the transition function $\varphi(\mathbf{0}, \mathbf{0}) = 0$. It is natural to add a similar restriction on the broadcasting function:

$$\psi(\mathbf{0}, \nu) = 0,$$

i.e. an inactive cell that doesn't have active neighbors cannot broadcast nonzero signals.

3.3. Partial definiteness of the global transition function

In equation (1) the value of locator $s_i(\alpha)$ is defined as a sum of the infinite number of terms by the integer points of the solid angle where a semigroup operation is used as an addition. An infinite sum is understood here in the usual sense (as the limit of partial sums) with the clarification that a discrete topology is introduced on the set E. In this case, for the series to converge, it is necessary that starting from some moment, the partial sums are equal to a constant, which is the sum of the series. This sum can be undefined if the sum involves an infinite number of nonzero terms. In the general case, the value of the locator is a partially defined function. Hence the global transition function of the CA with locators is also partially defined. However, even here a proof of correctness is required, namely, we need to prove that the convergence of the series (1) and the value of the sum does not depend on the order of terms (in the case of numerical series, this is true only for absolutely convergent series).

Proposition 1. *Let $(E,+)$ be a commutative semigroup with discrete topology. Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of elements E, $\{y_j\}_{j=1}^{\infty}$ be its permutation $(y_j = x_{i_j})$. Then if one of the series $\sum_{j=1}^{\infty} x_j$ and $\sum_{j=1}^{\infty} y_j$ converges, then the second also converges and their sums coincide.*

Доказательство. The proof goes by way of contradiction. Without loss of generality, assume that $\sum_{j=1}^{\infty} y_j = a$, and the series $\sum_{j=1}^{\infty} x_j$ either diverges or its sum is not equal to a. This means that in the sequence of partial sums $(X_n)_{n=1}^{\infty}$, $X_n = \sum_{j=1}^{n} x_j$ there is an infinite number of terms not equal to a. Since the first series converges, there exists N_0 such that for all $n \geq N_0$ the partial sum $Y_n = \sum_{j=1}^{n} y_j$ is equal to a. This means that $a + y_j = a$ for all $j > N_0$.

We denote $K_n = \{j \mid i_j \leq n\}$. Take $N \geq \max_{j \leq N_0} i_j$ such that $X_N = b \neq a$. By construction $1, 2, ..., N_0 \in K_N$. Hence

$$b = X_N = \sum_{j=1}^{N} x_j = \sum_{k \in K_N} y_k = \sum_{k=1}^{N_0} y_j + \sum_{j > N_0, j \in K_N} y_j = a + \sum_{j > N_0, j \in K_n} y_j = a.$$

However $b \neq a$ by our assumption, and we have a contradiction. Hence $\sum_{j=1}^{\infty} x_n = a$, as required. ☐

A state of a CA with locators is called *finite* if there is only a finite number of active cells. Note that, taking into account the previous corrections, for finite states the global function is defined since only active cells can broadcast nonzero signals. However, consider such a CA with locators:

$$\sigma = (\mathbb{Z}, \{0,1\}, \varnothing, \{0,1\}, \max, \{\Omega\}, \max, \max),$$

where Ω corresponds to the locator that receives signals from all directions. Suppose at the first moment there is exactly one cell is in state 1, and thus the state is finite. Then this call broadcasts signal 1 and all the cells at the second moment receive signal $\max(0,1) = 1$, hence they go to state 1 at the third moment. Therefore, the state at the third moment is not finite. If we take $\oplus$ instead of max as a semigroup operation, then the functioning at the first two moments will be the same, and at the third moment, the transition function will not be defined.

4. Interesting classes of CA with locators

4.1. Classes solving the problem of partial definiteness of the transition function

Taking into account the example from the Section 3.3, it is important to find classes of CA with locators $(\mathbb{Z}, Q, V, E, +, L, \varphi, \psi)$, where the definiteness of the global transition function is guaranteed at any moment of time for some class of initial conditions.

4.1.1. Idempotent monoid

Consider the case when the monoid $(E, +)$ is idempotent (is a semilattice), that is, $x + x = x$ for all $x \in E$. In this case, the sum of an infinite number of terms depends only on the set of terms present in the sum, and thus reduces to a finite sum. Therefore, we have the following statement.

Proposition 2. *If the monoid $(E, +)$ is idempotent, then the global transition function of a CA with locators $\sigma = (\mathbb{Z}, Q, V, E, +, L, \varphi, \psi)$ is defined everywhere.*

For example, if E is a linearly ordered set, then $(E, \max)$ is an idempotent monoid with neutral element $\min E$.

4.1.2. Finite CA with locators

We will say that the CA with locators σ is *finite* if for any finite state S of σ the next state $\Phi(S)$ is also finite.

Proposition 3 (Sufficient condition for the finiteness of a CA with locators). *Let* $\sigma = (\mathbb{Z}^n, Q, V, E, +, \{\nu_1, ..., \nu_m\}, \varphi, \psi)$ *be a CA with locators satisfying the following condition:*

$$\textit{if } \varphi(\vec{\mathbf{0}}, (e_1, ..., e_m)) \neq 0, \ \textit{then} \bigcap_{i: e_i \neq 0} \nu_i = \varnothing.$$

Then σ *is finite.*

In this statement it is important that we exclude vertex from the solid angle, otherwise, the intersection of the solid angles ν_i would always contain the origin.

Доказательство. Consider an arbitrary finite state s. Let A be a set of all cells that are either active itself or have active neighbors, r be the maximum Euclidean distance between elements of A.

Suppose the state $\Phi(s)$ is not finite. In this case, there is an infinite set M of cells that was not active and didn't have active neighbors in the finite state s and that became active in the state $\Phi(s)$. For each cell x from M consider the set of its active locators $a(x)$ and choose such a set of locators $L' \subset L$ that occurs infinitely many times among $a(x)$ for $x \in M$. Let $M' = \{x \in M : a(x) = L'\}$.

Without loss of generality we assume that $L' = \nu_1, ..., \nu_k$. From the statement condition we have $\bigcup_{j=1}^{k} \nu_j = \varnothing$.

Let S be the unit sphere in $\mathbb{R}^n$, $P = \prod_{j=1}^{k} (\nu_{i_j} \cap S)$. Let us show that

$$\hat{d} := \inf_{p \in P} \max_{1 \leq j, j' \leq k} \|p_j - p_{j'}\| > 0, \tag{4}$$

where $\| \cdot \|$ is the Euclidean norm.

Note that each set $\nu_{i_j} \cap S$ is compact, therefore, their product P is also a compact set. Hence continuous function $d(p) := \max_{j \neq j'} \|p_j - p_{j'}\|$ reaches its minimum on the compact set P. Suppose, this minimum is 0. Then there exist $p \in P$, $p_j \in \nu_j$ such that $p_j = p_{j'}$ for all $1 \leq j, j' \leq k$, that is, $p_1 = ... = p_k \in \bigcap_{j=1}^{k} \nu_j = \varnothing$, and we obtain a contradiction. Hence (4) is satisfied.

Since the set M' is infinite, there exists an element $x \in M'$ located at a distance $D > r/d$ from the set A. Since the cell x has the locators $\nu_1, ..., \nu_k$ active, there exist elements $y_1, ..., y_k \in A$ such that $v_j = y_j - x \in \nu_j$. Put

$p_j = \frac{v_j}{\|v_j\|}$. Then for any $1 \le i, j \le k$ the following holds:

$$\begin{aligned}\|p_i - p_j\|^2 = \|p_i\|^2 + \|p_j\|^2 - 2(p_i, p_j) = 2 - 2\frac{(v_i, v_j)}{\|v_i\|\|v_j\|} \le \\ \le \frac{\|v_i\|}{\|v_j\|} + \frac{\|v_j\|}{\|v_i\|} - 2\frac{(v_i, v_j)}{\|v_i\|\|v_j\|} = \frac{\|v_i - v_j\|^2}{\|v_i\|\|v_j\|} \le \\ \le \frac{\|v_i - v_j\|^2}{D^2} = \frac{\|y_i - y_j\|^2}{D^2} \le \frac{r^2}{D^2} < d^2.\end{aligned}$$

Thus, $\max_{1 \le i,j \le k} \|p_i - p_j\| < d$. On the other hand, $p_j \in \nu_j \cap S$, that is, $p = (p_1, ..., p_k) \in P$ which contradicts (4). Hence, our assumption is wrong, and the state $\Phi(s)$ is finite which completes the proof. □

4.2. Class with simple physical implementation

It is most natural to imagine the implementation of a CA with locators as a chip. The broadcasting should be implemented by some device that "sums up" an unlimited number of electrical signals. Such a device can consist of the following elements: a conductor connected to all cell outputs to be summed; an amplifier with the input connected to the conductor and output connected to the locator inputs of all the cells. Thus, if one of the cells emits a signal, this signal will be amplified and signal 1 will come to the locators of all cells. If all the cells emit 0, then 0 will come to the locators of all cells as well. In such a way we can implement the operation max from an unlimited number of arguments taking values from the set $\{0, 1\}$.

However, for a CA with locators, it is required to be able to calculate $\max_{i \ne j} a_j$ for all $i = 1, ..., m$. Note that

$$\max_{j \ne i} a_j = \min\Big(\sum_{j \ne i} a_j, 1\Big) = \min\Big(\min\Big(\sum_{j=1}^{m} a_j, 2\Big) - a_i, 1\Big).$$

it is also possible to implement operation $M_2(a_1, ..., a_n) = \min(\sum_{j=1}^{m}, 2)$, but more difficult than the operation max. For example, this can be done as follows. Each input representing an operation argument equal to 1 outputs a limited current to the wire connecting all the arguments and connected to the neutral wire through a resistor. Depending on the number of inputs equal to 1, there would be different voltages on the connecting conductor. The conductor itself can be connected to two comparators, of which one is triggered at voltage when at least one input is active, and the other is triggered when at voltage when at least 2 inputs are active. Using the results of these comparators, it is easy to obtain the value of the function M_2. Then, through the common wire, we can connect the result $s = M_2(a_1, ..., a_m)$ back to all cells, and calculate $\min(s - a_i, 1)$ in the i-th cell. Thus, the result in i-th cell is $\max_{j \neq i} a_j$ as required.

Using n copies of such a circuit, we can implement the Max operation on the set $\{0, 1\}^n$, which is a component-wise max operation:

$$\mathrm{Max}((a_1^1, ..., a_n^1), \cdots, (a_1^m, ..., a_n^m)) = (\max(a_1^1, ..., a_1^m), \cdots, \max(a_n^1, ..., a_n^m)).$$

Let show that arbitrary idempotent commutative monoid $(E, +)$ where $|E| = n < \infty$ can be implemented using operation Max and ordinary logic gates. To do this, we encode nonzero elements of E by the tuples $(1, 0, ..., 0)$, $(0, 1, 0, ..., 0)$, ..., $(0, ..., 0, 1) \in \{0, 1\}^{n-1}$, and we encode $0 \in E$ by the all-zero tuple. Let v be the described encoding function. For the set $E' = \{e_1, ..., e_m\} \subseteq E$, we define $\hat{v}(E') = \mathrm{Max}_{e \in E'} v(e)$. In the tuple $\hat{v}(E')$ ones occur at positions corresponding to nonzero elements of the set E'. Boolean operator $F : \hat{v}(E') \mapsto v(\sum_{e \in E'} e)$ can be implemented by a logic circuit. Using the idempotency of the monoid, for an arbitrary number of arguments we have

$$v\Big(\sum_{i \in I} e_i\Big) = v\Big(\sum_{e \in \{e_i | i \in I\}} e\Big) = F(\hat{v}(\{e_i \mid i \in I\})) = F(\mathop{\mathrm{Max}}_{i \in I} v(e_i)).$$

So, we proved that for any finite idempotent monoid it is possible to implement its semigroup operation from an unlimited number of elements using a fixed logic circuit and several conductors connected to all cells whose outputs are summed up.

This is exactly the class of monoids from Section 4.1.1, for which the global transition function is defined everywhere. The situation with the implementation of locators is worse. The conductor conducts in the same way in all directions. If we use diodes that pass current only in one direction, the depth of the circuit will immediately become linear in the number of arguments, and in this case, it is no longer possible to say that the broadcast is instant, thus the goal of using this model is lost. Therefore, only solid angles coinciding with subspaces can be implemented by the described method. For example, Ω is implemented if the outputs of all cells are connected with a plate. We can make a layer with many wires going in the same direction. Thus we can implement the locator $\{v, -v\}$, where v is the direction of the wires in this layer.

Implementation of other locators requires the use of some other physical principles that go beyond conventional circuit design.

References

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